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LAPLACE TRANSFORM TECHNIQUES FOR NONLINEAR SYSTEMS

by

John Epes Whitely

INTERNALLY DISTRIBUTED REPORT

United States Naval Postgraduate School

THESIS

LAPLACE TRANSFORM TECHNIQUES FOR NONLINEAR SYSTEMS

by

John Epes Whitely, Jr.

December 1970

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REPORT

Laplace Transform Techniques for Nonlinear Systems

by

John Epes Whitely, Jr.
Lieutenant Commander, United States Navy
B.S., United States Naval Academy, 1960

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

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ABSTRACT

Several techniques for finding approximate solutions to certain classes of nonlinear differential equations are investigated. The nonlinear systems evaluated are second order with quadratic and cubic terms, having driving functions and initial conditions. Pipes' technique is used to reduce the nonlinear differential equation to a system of linear differential equations. The Brady-Baycura technique makes use of nonlinear Laplace transforms to obtain the solution. The solutions are compared to the well known Runge-Kutta numerical method solution using the digital computer. Greater accuracy was found using the Brady-Baycura method, but the simplicity of Pipes' method makes it more attractive to the engineer.

TABLE OF CONTENTS

I. INTRODUCTION 5

II. METHODS 6

A. PIPES' METHOD 6

B. BRADY-BAYCURA METHOD 7

C. TOU-DOETSCH-PIPES-NOWACKI METHOD 8

III. APPLICATIONS 10

A. NONLINEAR SPRING 10

1. Solution by Pipes Reversion Method . . . 10

2. Solution by Brady-Baycura Transform Method 13

3. Solution by Tou-Doetsch-Pipes-Nowacki Method 15

4. Comparing the Methods 17

B. NONLINEAR SPRING WITH DAMPENING 17

1. Solution by Pipes Method 24

2. Solution by Brady-Baycura Method 27

3. Comparing the Solutions 30

C. NONLINEAR SPRING WITH CUBIC TERM 30

1. Solution by Pipes Technique 38

2. Solution by Brady-Baycura Method 42

3. Comparison of Solutions 43

D. NONLINEAR SPRING WITH DRIVING FUNCTION . . . 44

1. Solution by Pipes Method 51

2. Solution by Brady-Baycura Technique . . 52

3. Comparing the Solutions 56

E. NONLINEAR SPRING WITH CUBIC TERM AND FORCING FUNCTION 56

  1. Solution by Pipes Method 62
  2. Solution by Brady-Baycura Technique . . 63
  3. Comparing the Solutions 66

IV. CONCLUSIONS 72

APPENDIX A - Inverse Laplace Transform of $\frac{s}{(s^2 + 1)^3}$. 73

APPENDIX B - Evaluation of A_3 in Forced Quadratic System 74

APPENDIX C - Solution of Last Term of $F_2(s)$ by Brady-Baycura Method 76

BIBLIOGRAPHY 78

INITIAL DISTRIBUTION LIST 79

FORM DD 1473 80

I. INTRODUCTION

Although numerical methods such as Runge-Kutta have been widely used to solve nonlinear differential equations by means of digital computers, the engineer does not always have access to a computer. Practical techniques to assist the engineer in obtaining approximate solutions are needed.

Pipes (Ref. 1) developed a method of solving nonlinear differential equations by a reversion technique quite akin to algebraic series reversion. The technique uses Laplace transforms extensively in obtaining the approximate solution. Baycura and Brady (Refs. 2 and 3) have proposed and examined nonlinear Laplace transforms to obtain solutions to nonlinear differential equations. Tou, Doetsch, Pipes and Nowacki (Refs. 4, 5, 6 and 7) have proposed an iterative method using Laplace transforms in each iteration. Flake (Ref. 8) has proposed a cumbersome method using Volterra series and special transforms.

The techniques were investigated and applied to driven second order systems with initial conditions. The systems were restricted to known stable systems, and with driving functions far from resonance. The method of Flake was found to be so impractical that it is only mentioned here for reference.

II. METHODS

The various methods of obtaining approximate solutions to nonlinear differential equations were investigated.

A. PIPES' METHOD

Pipes has postulated a general nonlinear differential equation of the form

$$a_1 x + a_2 x^2 + \dots a_i x^i + \dots a_n x^n = k\,\phi(t) \tag{2.1.1}$$

where t is the independent variable, x is the dependent variable, k is a constant, $\phi(t)$ is a driving function, and the a_i are functions of the differential operator D, where

$$D = d/dt \tag{2.1.2}$$

Consider the special case of equation (2.1.1) where

$$a_1 = \sum_{j=0}^{n} b_j \frac{d^j}{dt^j}; \quad j = 0, 1, \dots, n. \tag{2.1.3}$$

and the a_i are constant coefficients for i = 2, 3, ..., n. Equation (2.1.1) becomes

$$\left(b_0 + b_1 \frac{d}{dt} + b_2 \frac{d^2}{dt^2} + \dots + b_n \frac{d^n}{dt^n}\right) x + a_2 x^2 + \dots + a_n x^n = k\,\phi(t) \tag{2.1.4}$$

or using the operator notation

$$(b_0 + b_1 D + b_2 D^2 + \dots + b_n D^n) x + a_2 x^2 + \dots + a_n x^n = k\,\phi(t) \tag{2.1.5}$$

Following Pipes, assume a series solution of the form

$$x(t) = A_1(t) k + A(t) k^2 + A(t) k^3 + \dots \tag{2.1.6}$$

By substituting equation (2.1.6) into (2.1.1) and equating coefficients of equal powers of k, the various $A_i(t)$ terms may be found. They are

$$A_1(t) = \phi(t)/a_1 \tag{2.1.7}$$

$$A_2(t) = -a_2 A_1^2(t)/a_1 \tag{2.1.8}$$

$$A_3(t) = -(1/a_1)\left[2a_2 A_1(t)A_2(t) + a_3 A_1^3(t)\right] \tag{2.1.9}$$

$$A_4(t) = -(1/a_1)\left[a_2\left(A_2^2(t) + 2A_1(t)A_3(t)\right) + 3a_3 A_1^2(t)A_2(t) + a_4 A_1^4(t)\right] \tag{2.1.10}$$

$$\begin{aligned} A_5(t) = -(1/a_1)\Big[&2a_2\left(A_1(t)A_4(t) + A_2(t)A_3(t)\right) + 3a_3\left(A_1(t)A_2^2(t) + A_1^2(t)A_3(t)\right) \\ &+ 4a_4 A_1^3(t)A_2(t) + a_5 A_1^5(t)\Big] \end{aligned} \tag{2.1.11}$$

Pipes (Ref. 1) lists additional terms. However, since a second or third approximation to the solution will generally be all that is desired, only the first five terms are given here.

A first order approximation can be found by solving the linear differential equation (2.1.7). Since (2.1.7) is a linear differential equation, it can be solved by the usual techniques of integration or Laplace transforms, applying initial conditions. The remaining $A_i(t)$ are solved in like manner, except the initial conditions are zero.

B. BRADY-BAYCURA METHOD

Brady and Baycura have shown that for exponential functions and for small values of time the following approximate expression holds

$$\mathcal{L}\left[x^n(t)\right] \approx s^{n-1} X^n(s) \tag{2.2.1}$$

Since the Laplace transform of Ø(t) is known, all transforms form the terms of equation (2.1.3) are known and thus (2.1.3) can be solved for X(s).

C. TOU-DOETSCH-PIPES-NOWACKI METHOD

This method probably originated with Pipes, but was arrived at independently by the others.

Assume the nonlinear system differential equation of the form

$$\left[a_n D^n + a_{n-1} D^{n-1} + \ldots + a_i D^i + \ldots + a_0\right] y + f(t, y, \dot{y}, \ldots) = g(t) \tag{2.3.1}$$

Taking the Laplace transforms of both sides of equation (2.3.1) yields

$$Y(s) \sum_{k=0}^{n} a_k s^k - Q(s) + \mathcal{L}\left\{f(t, y, \dot{y}, \ldots)\right\} = G(s) \tag{2.3.2}$$

where Q(s) is the set of initial conditions.

In the first approximation assume the nonlinear term $\mathcal{L}\left[f(t, y, \dot{y}, \ldots)\right] = 0$, then

$$Y_1(s) = \frac{Q(s) + G(s)}{\sum_{k=0}^{n} a_k s^k} \tag{2.3.3}$$

The inverse Laplace transform of $Y_1(s)$ yields the time domain solution of the first approximation.

The nonlinear term is included in the second approximation.

$$Y_2(s) = \frac{Q(s) + G(s) + \mathcal{L}\left[f(t, y_1, \dot{y}_1, \ldots)\right]}{\sum_{k=0}^{n} a_k s^k} \tag{2.3.4}$$

where $f(t, y_1, y_1, \ldots)$ is a function of the first approximation of $y_1(t)$.

Successive approximations are found by continuing the same procedure.

III. APPLICATIONS

Nonlinear systems investigated were limited to second order with initial conditions and extended to systems with forcing functions.

A. NONLINEAR SPRING

Consider a simple spring-mass system described by

$$m\frac{d^2x}{dt^2} + F(x) = 0 \tag{3.1.1}$$

The system is undamped, with an initial displacement of $x(0) = x_o$, and initial velocity set to zero. $F(x)$ is the spring force, and for this example, let its characteristic be

$$F(x) = kx + bx^2 \tag{3.1.2}$$

Letting all constants be unity, the system equation is nonlinear

$$\frac{d^2x}{dt^2} + x + x^2 = 0 \tag{3.1.3}$$

$$\left.\begin{aligned} x(0) &= x_o \\ \dot{x}(0) &= 0 \end{aligned}\right\} \tag{3.1.4}$$

1. Solution by Pipes Reversion Method

Using the method outlined in Chapter II, Section A, equation (3.1.4) can be rewritten as

$$(D^2+1)x + x^2 = 0 \tag{3.1.5}$$

The a_i coefficients are seen to be

$$\left.\begin{aligned} a_1 &= D^2+1 \\ a_2 &= 1 \\ k &= 1 \\ \emptyset(t) &= 0 \end{aligned}\right\} \tag{3.1.6}$$

The individual A_i coefficients are then computed using the formulas (2.1.8), (2.1.9) and (2.1.10):

$$A_1(t) = \phi(t)/a_1 \tag{2.1.7}$$

$$(D^2+1)A_1 = 0 \tag{3.1.7}$$

Taking the Laplace transform of equation (3.1.3) and impressing initial conditions yields:

$$A_1(s) = \frac{s x_0}{s^2+1} \tag{3.1.8}$$

This well known transform gives

$$A_1(t) = x_0 \cos t \tag{3.1.9}$$

As expected, the system is oscillatory.

The next term $A_2(t)$ is found from equation (2.1.8)

$$A_2(t) = -\frac{a_2 A_1^2}{a_1} = \frac{-x_0^2 \cos^2 t}{D^2+1} \tag{3.1.10}$$

$A_2(t)$ and all subsequent $A_i(t)$ are determined by the condition that they have zero initial conditions.

Using the identity

$$\cos^2 t = \frac{1}{2}(1+\cos 2t) \qquad (3.1.11)$$

and taking Laplace transforms of both sides of equation (3.1.10)

$$A_2(s) = -\frac{x_0^2}{2}\left[\frac{1}{s(s^2+1)} + \frac{s}{(s^2+1)(s^2+4)}\right] \qquad (3.1.12)$$

Both terms can be expanded into partial fractions:

$$A_2(s) = -\frac{x_0^2}{2}\left[\frac{1}{s} - \frac{s}{s^2+1} + \frac{\frac{1}{3}s}{s^2+1} - \frac{\frac{1}{3}s}{s^2+4}\right] \qquad (3.1.13)$$

Taking inverse Laplace transforms:

$$A_2(t) = -\frac{x_0^2}{2}\left[1 - \frac{2}{3}\cos t - \frac{1}{3}\cos 2t\right] \qquad (3.1.14)$$

$A_3(t)$ is found in a similar manner, and the solution is given by

$$x(t) = A_1(t)k + A_2(t)k^2 + A_3(t)k^3 + \ldots \qquad (2.1.6)$$

$$x(t) \approx x_0\cos t - \frac{x_0^2}{2}\left[1 - \frac{2}{3}\cos t - \frac{1}{3}\cos 2t\right]$$

$$+ \frac{x_0^3}{3}\left[\frac{5}{2}t\sin t - 1 + \frac{29}{48}\cos t + \frac{1}{3}\cos 2t + \frac{1}{16}\cos 3t\right] + \ldots \qquad (3.1.15)$$

In equation (3.1.15) it is observed that the first secular[1] term occurs in the third approximation.

---

[1] Secular terms are those involving t sin t, t cos t, t^2 sin t, ..., etc.

2. Solution by Brady-Baycura Transform Method

Taking the Laplace transform of equation (3.1.3) and applying initial conditions gives

$$s^2X(s) - sx_0 + X(s) + sX^2(s) = 0 \tag{3.1.16}$$

In the usual manner, equation (3.1.16) is solved as a quadratic. After rearranging terms

$$X(s) = \frac{s^2+1}{2s}\left[-1 \pm \left(1 + \frac{4x_0s^2}{(s^2+1)^2}\right)^{1/2}\right] \tag{3.1.17}$$

Neglecting the negative radical as physically impossible, the positive radical can be expanded as a binomial

$$X(s) = \frac{sx_0}{s^2+1} - \frac{s^3x_0^2}{(s^2+1)^3} + \frac{2x_0^3s^5}{(s^2+1)^5} \tag{3.1.18}$$

The first term is just x_o cos t; however $\mathcal{L}^{-1}\left[\frac{s^3}{(s^2+1)^3}\right]$ is not one of the common transforms. Reference 10 lists the following transform:

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+\omega_0^2)^n}\right] = \frac{\pi^{1/2}\, t^{n-1/2} J_{n-3/2}(\omega_0 t)}{2^{n-1/2}\,\Gamma(n)\,\omega_0^{\,n-3/2}} \tag{3.1.19}$$

where Jn-3/2(w_ot) is the half order Bessel function.

Further it is known from the theory of Laplace transforms that

$$\mathcal{L}^{-1}\left[s^n G(s)\right] = f^{(n)}(t) + \sum_{k=0}^{n-1} f^{(k)}(0)\, s^{n-1-k}, \quad n = 1, 2, 3, \ldots \tag{3.1.20}$$

Since $f^{(k)}(0) = 0$ for $k = 0, 1, 2, \ldots, n-1$, the second term can be found:

letting $\omega_0 = 1$

$$J_{3/2}(t) = \left(\frac{2}{\pi t}\right)^{1/2}\left(\frac{\sin t}{t} - \cos t\right) \tag{3.1.21}$$

and after carrying out the differentiations of equation (3.1.20), the second term of equation (3.1.18) becomes

$$-\frac{x_0^2}{8}\left[3t\sin t + t^2\cos t\right] \tag{3.1.22}$$

Repeated application of the method will yield the third term.

The approximate solution to three terms is the truncated series

$$x(t) \approx x_0\cos t - \frac{x_0^2}{8}\left[3t\sin t + t^2\cos t\right] + \frac{x_0^3}{32}\left[\frac{5}{2}\left(t\sin t - t^2\cos t\right) + \frac{5}{3}t^3\sin t + \frac{t^4}{6}\cos t\right] + \cdots \tag{3.1.23}$$

3. Solution by Tou-Doetsch-Pipes-Nowacki Method

Putting equation (3.1.3) in the form of (2.3.1)

$$a_2\ddot{x} + a_0 x + x^2 = 0 \qquad (3.1.24)$$

where

$$\left.\begin{array}{l} a_0 = 1 \\ a_1 = 0 \\ a_2 = 0 \\ f(t,y,\dot{y},\ldots) = x^2 \\ g(t) = 0 \end{array}\right\} \qquad (3.1.25)$$

Taking Laplace transforms of both sides

$$Y(s)\sum_{k=0}^{n} a_k s^k - Q(s) + \mathcal{L}\{f(t,y,\dot{y},\ldots)\} = G(s) \qquad (3.1.26)$$

$$X(s)[s^2+1] - sx_0 + \mathcal{L}[x^2] = 0 \qquad (3.1.27)$$

Solving for $X_1(s)$ as in equation (2.3.2)

$$X_1(s) = \frac{Q(s) + G(s)}{\sum_{k=0}^{n} a_k s^k} = \frac{x_0 s}{s^2+1} \qquad (3.1.28)$$

Then taking inverse Laplace transforms yields the first approximate solution:

$$X_1(t) = x_0 \cos t \qquad (3.1.29)$$

The second approximation is found by applying equation (2.3.3)

$$X_2(s) = \frac{Q(s) + G(s) - \mathcal{L}\{f(t, x_1, \dot{x}_1, \ldots)\}}{\sum_{k=0}^{n} a_k s^k} \qquad (3.1.30)$$

$$X_2(s) = \frac{x_0 s}{s^2+1} - \frac{1}{s^2+1}\left[\mathcal{L}\{x_0^2 \cos^2 t\}\right] \qquad (3.1.31)$$

Using the identity of equation (3.1.11), equation (3.1.31) becomes

$$X_2(s) = \frac{x_0 s}{s^2+1} - \frac{x_0^2}{2(s^2+1)}\left[\frac{1}{s} + \frac{1}{s^2+4}\right] \qquad (3.1.32)$$

Taking the inverse Laplace transform gives the second approximation as

$$x_2(t) = x_0 \cos t - \frac{x_0^2}{2}\left[1 - \frac{2}{3}\cos t - \frac{1}{3}\cos 2t\right] \qquad (3.1.33)$$

Repeating the method gives $x_3(t)$ and the result is the same as equation (3.1.15).

While this method appears to be different than Pipes method of Chapter II, Section A, consideration of the manipulation of the solution shows the two methods to be equivalent. Since the method of Chapter II, Section A is less cumbersom, it will be used hereafter.

4. Comparing the Methods

Figures 1 and 2 are phase portraits of the solution as given by the IBM 360 computer. Runge-Kutta integration technique was used to obtain the solution. From these plots, it is clear that the system is oscillatory for $x_o \leq 0.5$ units and unstable otherwise.

Figures 3, 4 and 5 compare the solutions by Pipes method and the Brady-Baycura technique for various initial conditions.

The graphs demonstrate the relative accuracy of the approximations. The Brady-Baycura technic gives a better approximation in each instance; however, Pipes method is as accurate for small initial conditions. The difference is caused by the location of the secular terms in the solution. For both methods, however, the approximations are of little value for times greater than one unit.

No attempt has been made to suppress the secular terms which cause the approximate solution to diverge. It is generally known that subharmonic resonance terms (submultiples of the natural frequency) exist in these nonlinear systems; they do not occur in this solution. The higher frequency jump resonance terms do occur.

B. NONLINEAR SPRING WITH DAMPENING

The system of example 3.1 was modified to include frictional dampening. The resulting system equation takes on a velocity term. Letting all coefficients be unity, except the velocity term, the system equation becomes

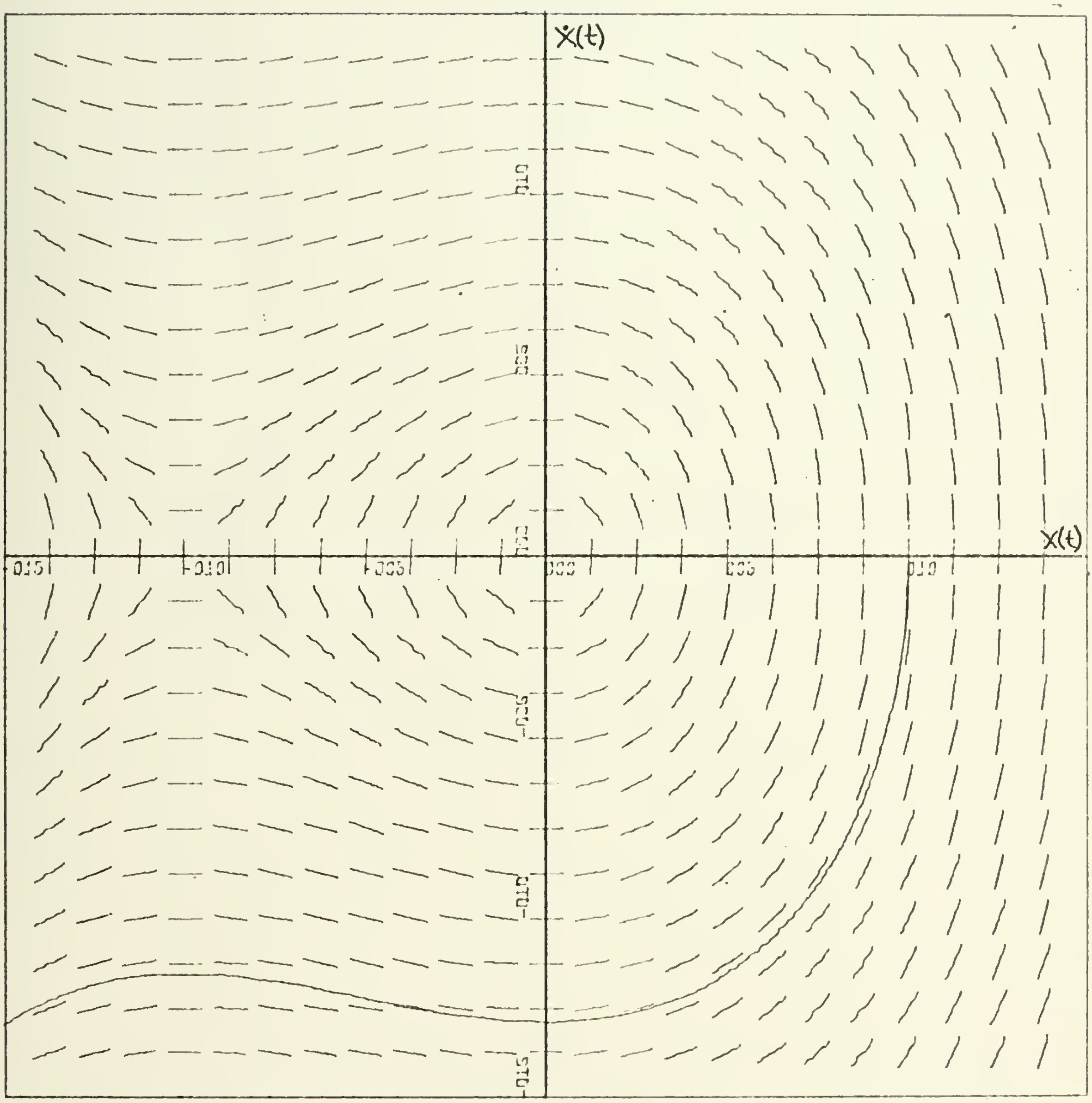

FIGURE 1. Phase portrait of $\ddot{x} + x + x^2 = 0$, $x_o = 1.0$

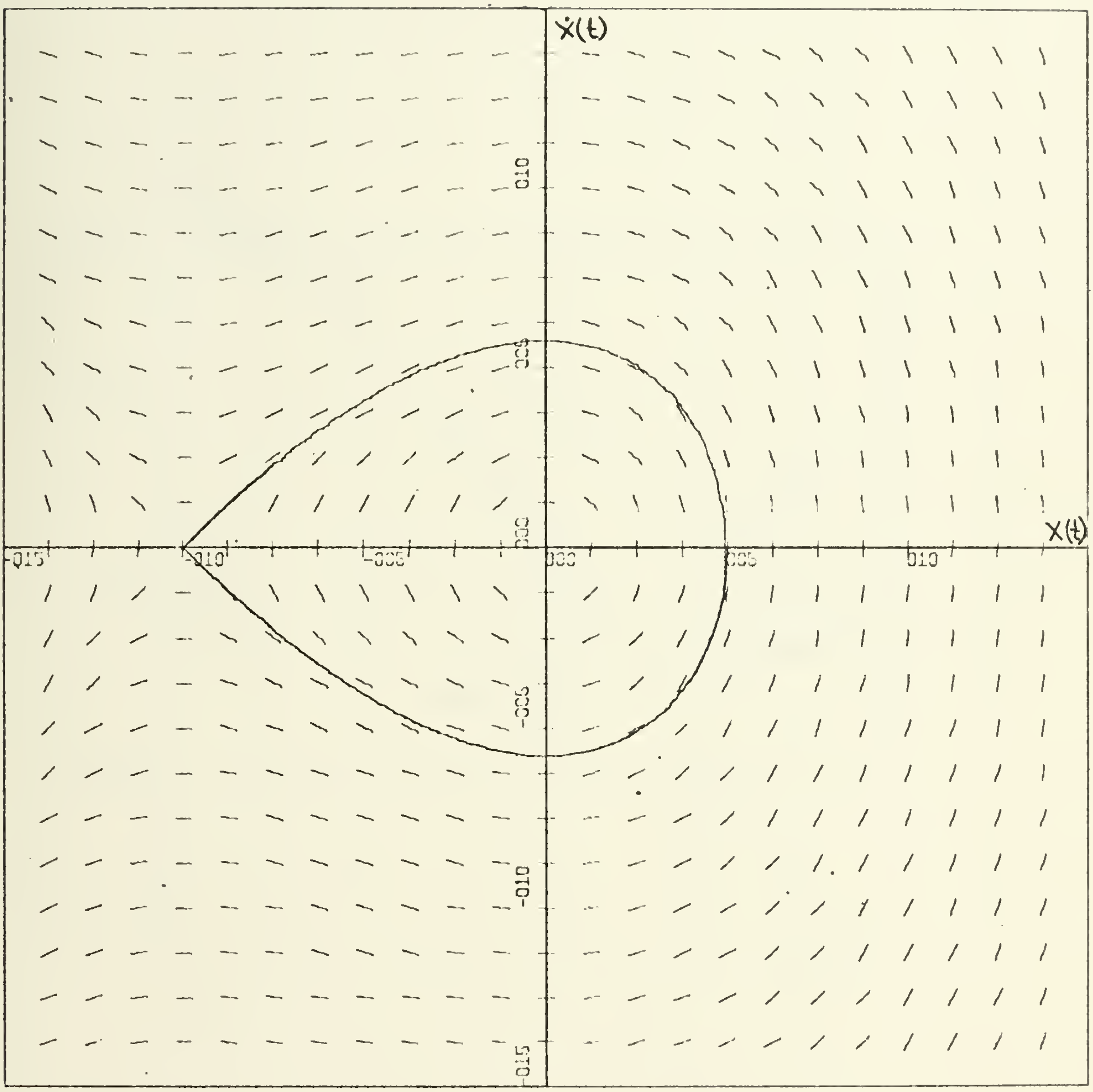

FIGURE 2. Phase portrait of $\ddot{x} + x + x^2 = 0$, $x_o = 0.5$

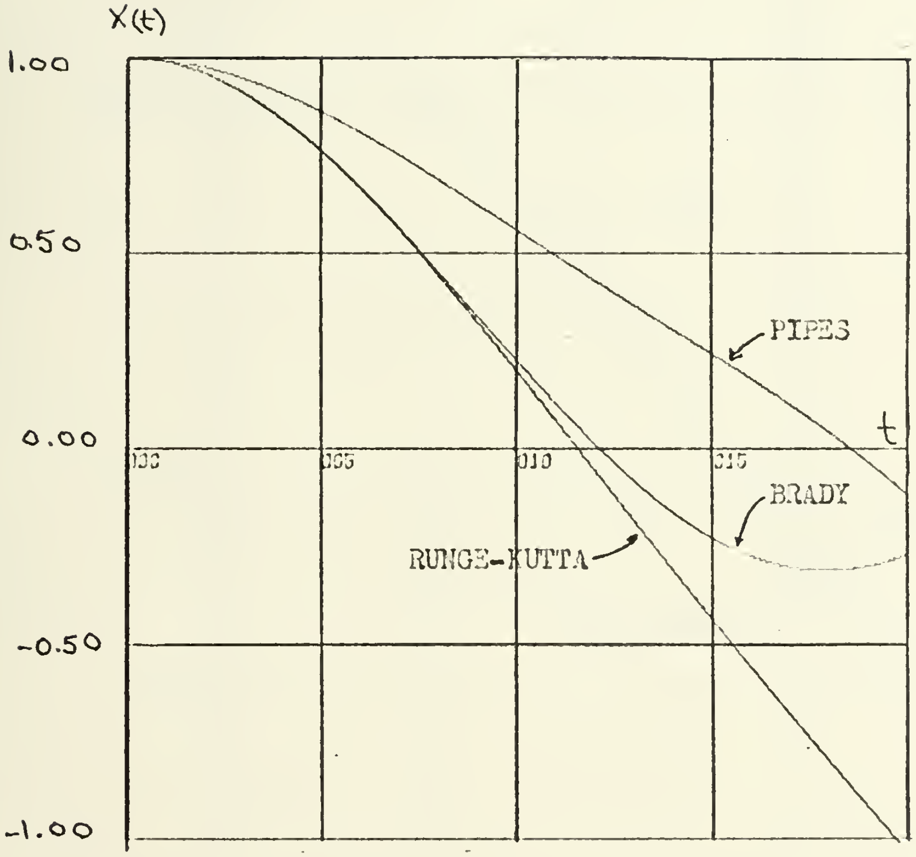

FIGURE 3. Time solution of $\ddot{x} + x + x^2 = 0$, $x_o = 1.0$

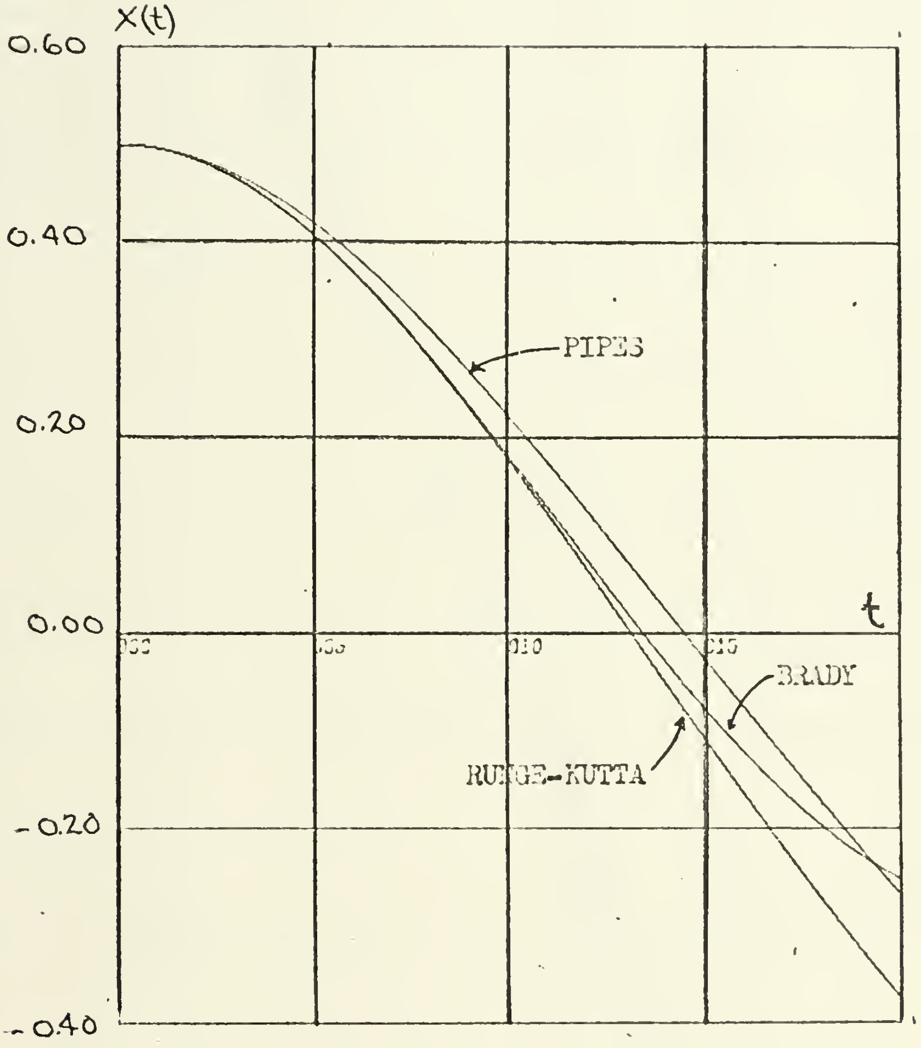

FIGURE 4. Time solution of $\ddot{x} + x + x^2 = 0$, $x_o = 0.5$

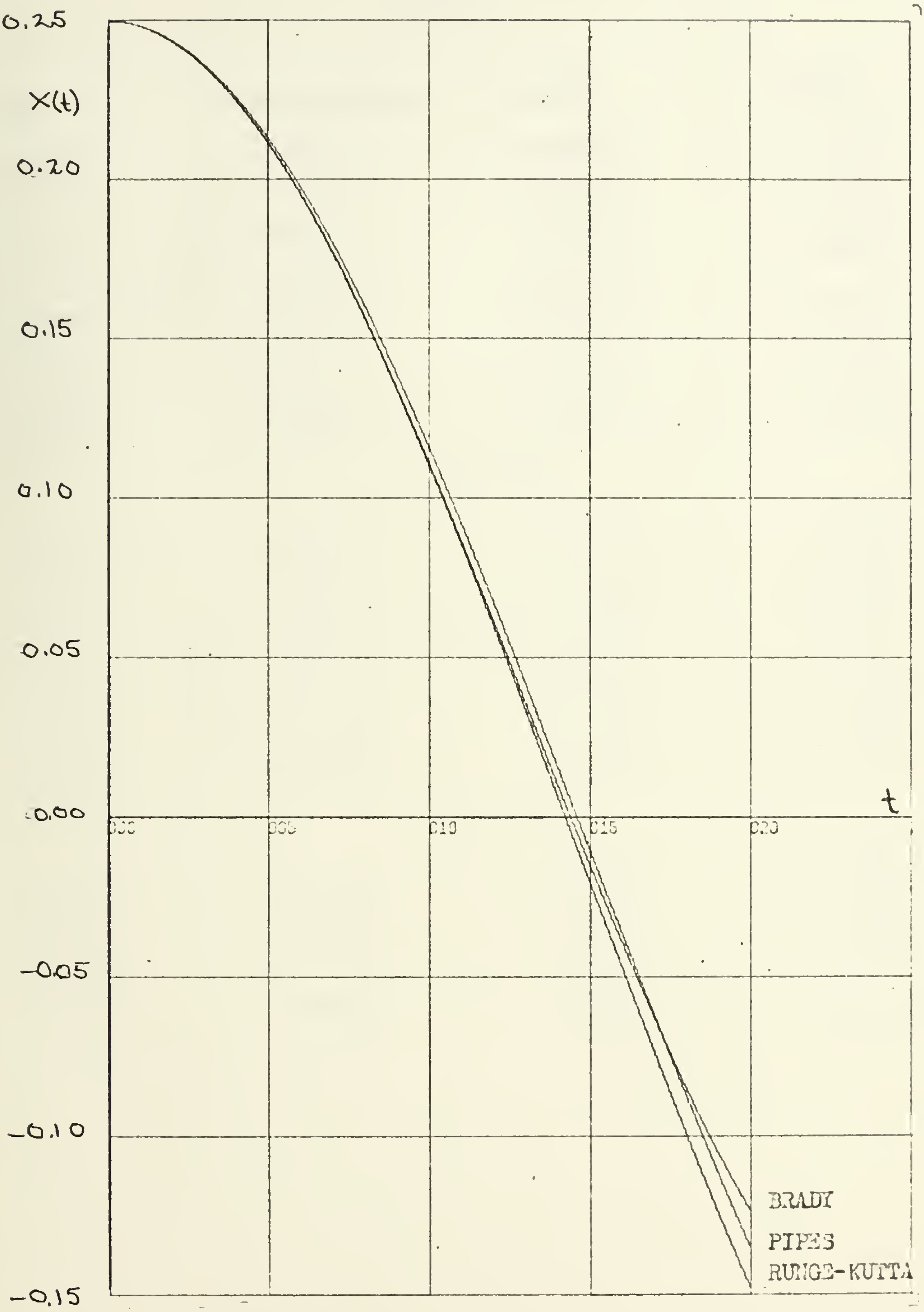

FIGURE 5. Time solution of $\ddot{x} + x + x^2 = 0$, $x_o = 0.25$

TABLE I

$x_o = 1$

Time	Runge-Kutta	Brady	Pipes
0.000	1.00000	1.00000	1.00000
0.050	0.99750	0.99750	0.99854
0.100	0.99002	0.99002	0.99418
0.150	0.97762	0.97762	0.98697
0.200	0.96039	0.96038	0.96424
0.500	0.76476	0.76469	0.86483
1.000	0.20173	0.22176	0.56088
1.500	-0.43604	-0.23227	0.24276
	$x_o = 0.50$		
0.000	0.50000	0.50000	0.50000
0.050	0.49906	0.49906	0.49919
0.100	0.49626	0.49626	0.49678
0.300	0.46675	0.46674	0.47137
0.500	0.41002	0.41002	0.42252
1.000	0.17967	0.18245	0.22407
1.5000	-0.10512	-0.075097	-0.02327
	$x_o = 0.25$		
0.000	0.25000	0.25000	0.25000
0.050	0.24961	0.24961	0.24963
0.1000	0.24844	0.24844	0.24850
0.500	0.21213	0.21213	0.21369
1.000	0.11163	0.11214	0.11715

$$\frac{d^2x}{dt^2} + C\frac{dx}{dt} + x + x^2 = 0 \tag{3.2.1}$$

$$\left.\begin{aligned} x(0) &= x_0 \\ \dot{x}(0) &= 0 \end{aligned}\right\} \tag{3.2.2}$$

The investigation included solving the system equation by Pipes and Brady-Baycura techniques. The final solution is found by letting the coefficient C approach zero in both methods.

1. Solution by Pipes Method

Using operator notation, the a_i's are

$$\left.\begin{aligned} a_1 &= D^2 + CD + 1 \\ a_2 &= 1 \\ \phi(t) &= 0 \\ k &= 1 \end{aligned}\right\} \tag{3.2.3}$$

By letting $C = 2\beta$, the algebra is simplified and the same result will obtain by letting 2β go to zero.

Proceeding with the solution $A_1 = \phi / a_1$ and upon substituting and rearranging

$$(D^2 + CD + 1)A_1(t) = 0 \tag{3.2.4}$$

Taking Laplace transforms and impressing initial conditions, equation (3.2.4) becomes

$$(s^2 + cs + 1)A_1(s) = x_o(s+c) \tag{3.2.5}$$

or

$$A_1(s) = \frac{x_o(s+c)}{(s+\beta-\sqrt{\beta^2-1})(s+\beta+\sqrt{\beta^2-1})} \tag{3.2.6}$$

Reference 11, formula 119, gives the inverse transform of equation (3.2.6) and

$$A_1(t) = \frac{x_o}{2\sqrt{\beta^2-1}}\Big[(c-\beta-\sqrt{\beta^2-1})\exp(-\beta+\sqrt{\beta^2-1})t \\ -(c-\beta-\sqrt{\beta^2-1})\exp(-\beta-\sqrt{\beta^2-1})t\Big] \tag{3.2.7}$$

letting 2β go to zero

$$A_1(t) = x_o\left[\frac{\exp(jt) + \exp(-jt)}{2}\right] \tag{3.2.8}$$

$$A_1(t) = x_o\cos t \tag{3.2.8a}$$

The solution for A_2, after a modicum of algebra becomes

$$A_2(t) = -\frac{x_o^2}{2}\left[1 - \frac{2}{3}\cos t - \frac{1}{3}\cos 2t\right] \tag{3.2.9}$$

and substituting $A_2(t)$ into equation (2.1.5), $A_3(t)$ becomes

$$A_3(t) = \frac{x_o^3}{D^2+CD+1}\left[\cos t - \frac{2}{3}\cos^2 t - \frac{1}{3}\cos t\cos 2t\right] \tag{3.2.10}$$

or

$$A_3(t) = \frac{x_0^3}{D^2+2\beta D+1}\left[-\frac{1}{3}+\frac{5}{6}\cos t-\frac{1}{3}\cos 2t-\frac{1}{6}\cos 3t\right] \tag{3.2.11}$$

Taking Laplace transforms, equation (3.2.11) becomes

$$\begin{aligned} A_3(s) = x_0^3\Bigg[& -\frac{1}{s(s^2+2\beta s+1)} + \frac{5/6\, s}{(s^2+1)(s^2+2\beta s+1)} \\ & - \frac{1/3\, s}{(s^2+4)(s^2+2\beta s+1)} - \frac{1/6\, s}{(s^2+9)(s^2+2\beta s+1)}\Bigg] \end{aligned} \tag{3.2.12}$$

All inverse transforms of the terms of equation (3.2.12), except the second, can be found in Ref. 10. The second term is evaluated by convolution as follows:

Let

$$Z_1(s) = \frac{s}{s^2+1} \tag{3.2.13}$$

$$Z_2(s) = \frac{1}{s^2+2\beta s+1} \tag{3.2.14}$$

then $z_1(t) = \cos t$ and $z_2(t) = \sin t$ when 2β goes to zero.

Convolving z_1 and z_2

$$\begin{aligned} Z_1(t) * Z_2(t) &= \int_0^t Z_1(\tau)\, Z_2(t-\tau)\, d\tau \\ &= \frac{t}{2}\sin t \end{aligned} \tag{3.2.15}$$

The approximate solution to three terms by Pipes technique is

$$x(t) \approx x_0 \cos t - \frac{x_0^2}{2}\left[1 - \frac{2}{3}\cos t - \frac{1}{3}\cos 2t\right]$$

$$+ \frac{x_0^3}{3}\left[\frac{5}{4} t \sin t - 3 + \frac{125}{48}\cos t - \frac{1}{3}\cos 2t - \frac{1}{16}\cos 3t\right] + \dots \qquad (3.2.16)$$

It is observed that the first secular term occurs in the third approximation.

2. Solution by Brady-Baycura Method

The solution proceeds by taking the Laplace transform of equation (3.2.1), and impressing the initial conditions. The nonlinear Laplace transform is used for the $x^2(t)$ term.

$$sX^2(s) + (s^2 + cs + 1)X(s) = x_0(s + c) \qquad (3.2.17)$$

By reversion of series technique, the left side of equation (3.2.17) can be made into a series by using the formulas of Ref. 9.

Let equation (3.2.17) have the form

$$x_0(s + c) = b_1 X(s) + b_2 X^2(s) + \dots = y \qquad (3.2.18)$$

where

$$\left.\begin{aligned} b_1 &= s^2 + cs + 1 \\ b_2 &= s \end{aligned}\right\} \qquad (3.2.19)$$

then by reversion of the series

$$X(s) = B_1 y + B_2 y^2 + B_3 y^3 + \ldots \tag{3.2.20}$$

where

$$y = x_0(s+c) \tag{3.2.21}$$

$$B_1 = \frac{1}{b_1} = \frac{1}{s^2+cs+1} \tag{3.2.22}$$

$$B_2 = -\frac{b_2}{b_1^3} = -\frac{s}{(s^2+cs+1)^3} \tag{3.2.23}$$

$$B_3 = \frac{1}{b_1^5}\left(2b_2^2 - b_1 b_3\right) = \frac{2s^2}{(s^2+cs+1)^5} \tag{3.2.24}$$

or letting $c = 2\beta$

$$X(s) = \frac{x_0(s+c)}{s^2+2\beta s+1} - \frac{s x_0^2 (s+c)^2}{(s^2+2\beta s+1)^3} + \frac{2x_0^3 s^2 (s+c)^3}{(s^2+2\beta s+1)^5} + \ldots \tag{3.2.25}$$

The first term of equation (3.2.25) is the same as equation (3.2.6), and is just x_o cos t, as in the Pipes solution.

Evaluation of the second term must be found by convolution:

Let

$$Z_1(s) = \frac{s}{s^2 + 2\beta + 1} \tag{3.2.26}$$

$$Z_2(s) = \frac{s + 2\beta}{s^2 + 2\beta + 1} \tag{3.2.27}$$

Taking inverse transforms, and letting 2β go to zero,

$$Z_1(t) = \cos t \tag{3.2.28}$$

$$Z_2(t) = \cos t \tag{3.2.29}$$

Then the inverse of the second term is the convolution

$$Z(t) = Z_1(t) * Z_2(t) * Z_2(t) \tag{3.2.30}$$

Again after a modicum of calculus, algebra and trigonometric substitutions, equation (3.2.30) reduces to just

$$Z(t) = \frac{1}{2}\left[\frac{t^2}{4}\cos t + t\sin t\right] \tag{3.2.31}$$

and to two terms, the Brady-Baycura technique yields

$$x(t) \approx x_0 \cos t - \frac{x_0^2}{2}\left[\frac{t^2}{4}\cos t + t\sin t\right] + \cdots \tag{3.2.32}$$

The technique has lost its desirability even at this point, since the inverse transforms are so difficult and time consuming to evaluate.

3. Comparing the Solutions

The phase portraits given in Figures 6, 7a and 7b show the apparent warping of the phase plane due to the inclusion of the velocity term. The instability of the system is seen to depend on both the value of C and x_o.

The Runge-Kutta solution is compared with the approximate solutions of equations (3.2.16) and (3.2.32) in Figures 8, 9 and 10. Table II lists selected values of the Runge-Kutta solution and the approximations.

It is clear from the table that neither approximations are good even at 0.5 time units. It is further obvious that for values of the coefficient $C \ll 1.0$, the system equation (3.2.1) will approach equation (3.1.3). Under the condition of C small, the techniques might have some value in approximating the transient state for small times (milliseconds).

C. NONLINEAR SPRING WITH CUBIC TERM

Consider again equation (3.1.1), but let

$$F(x) = kx + bx^3 \qquad (3.3.1)$$

Also let all coefficients be unity and the system have the same initial conditions as before. The nonlinear expression describing the system is

$$\frac{d^2x}{dt^2} + x + x^3 = 0 \qquad (3.3.2)$$

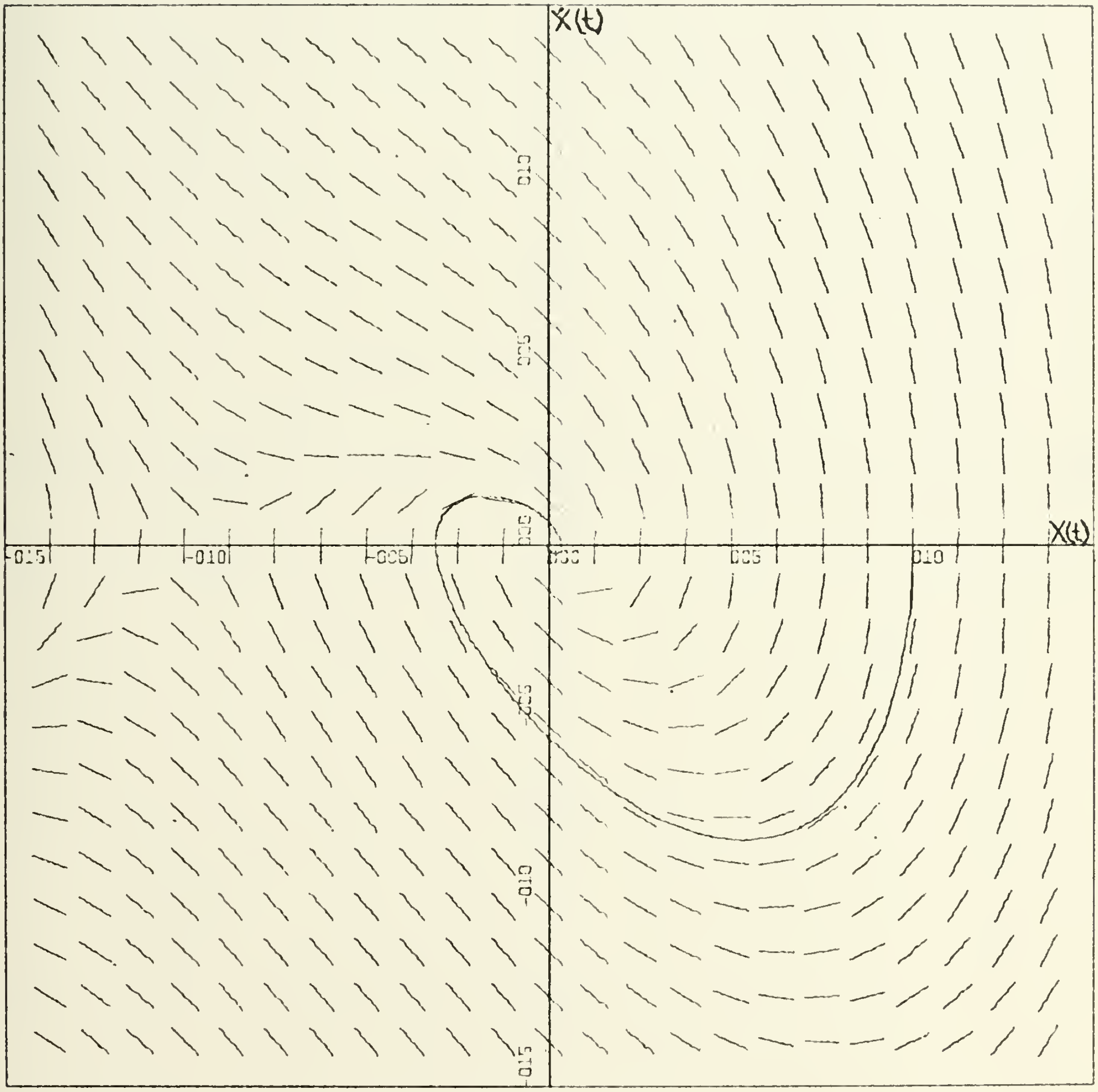

FIGURE 6. Phase portrait of $\ddot{x} + Cx + x + x^2 = 0$, $x_o = 1.0$

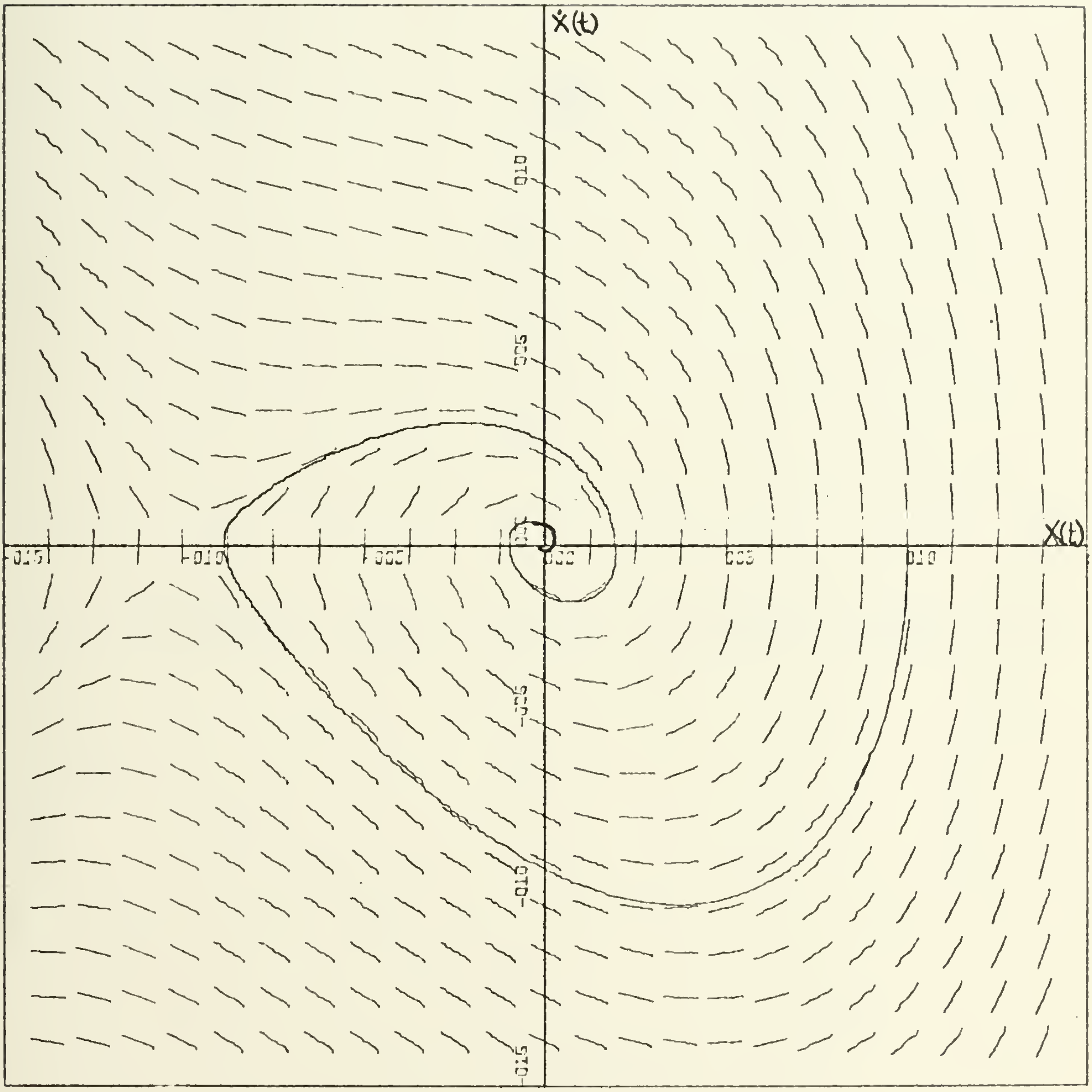

FIGURE 7a. Phase portrait of $\ddot{x} + 0.5x + x + x^2 = 0$, $x_o = 1.0$

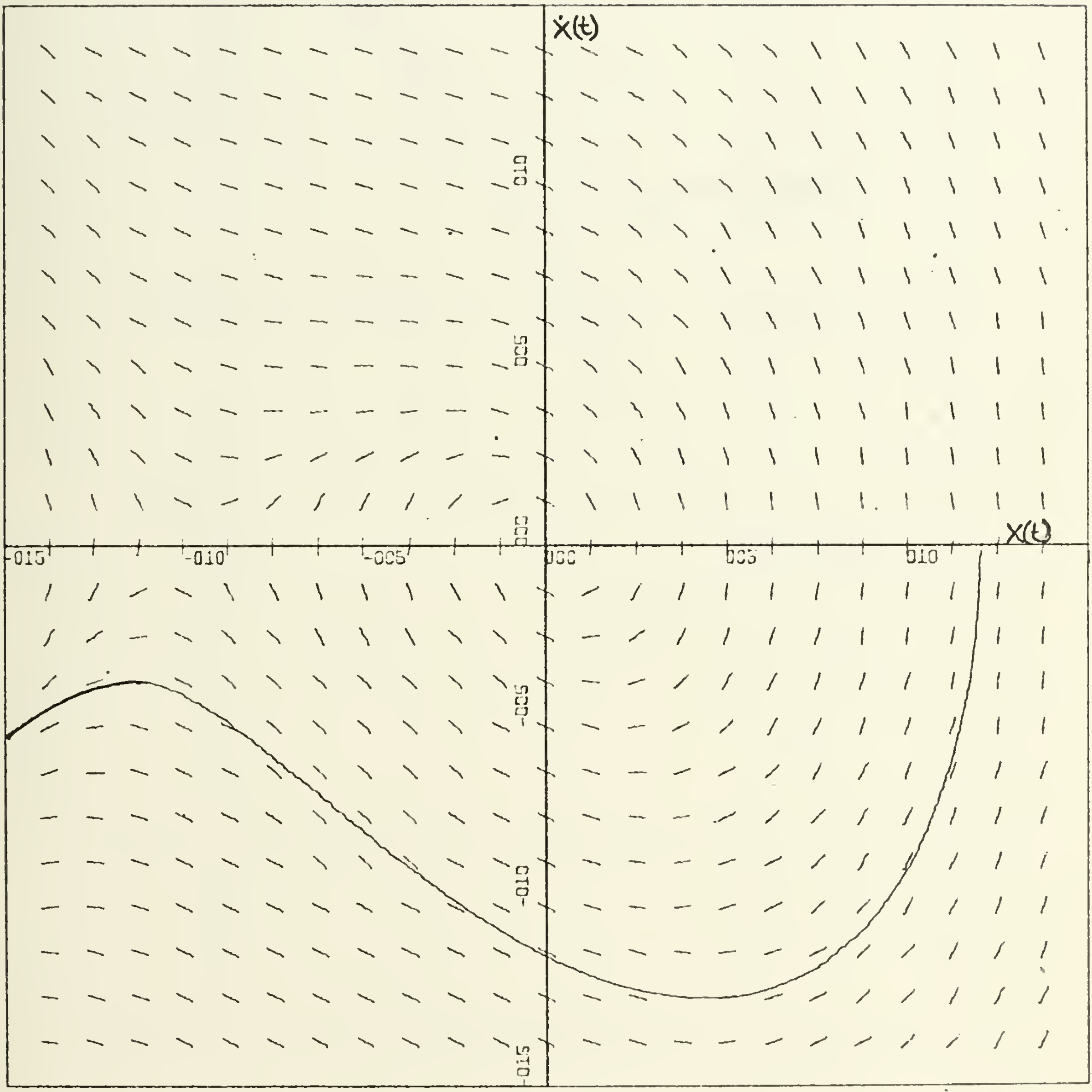

FIGURE 7b. Phase portrait of $\ddot{x} + 0.5x + x + x^2 = 0$, $x_o = 1.2$

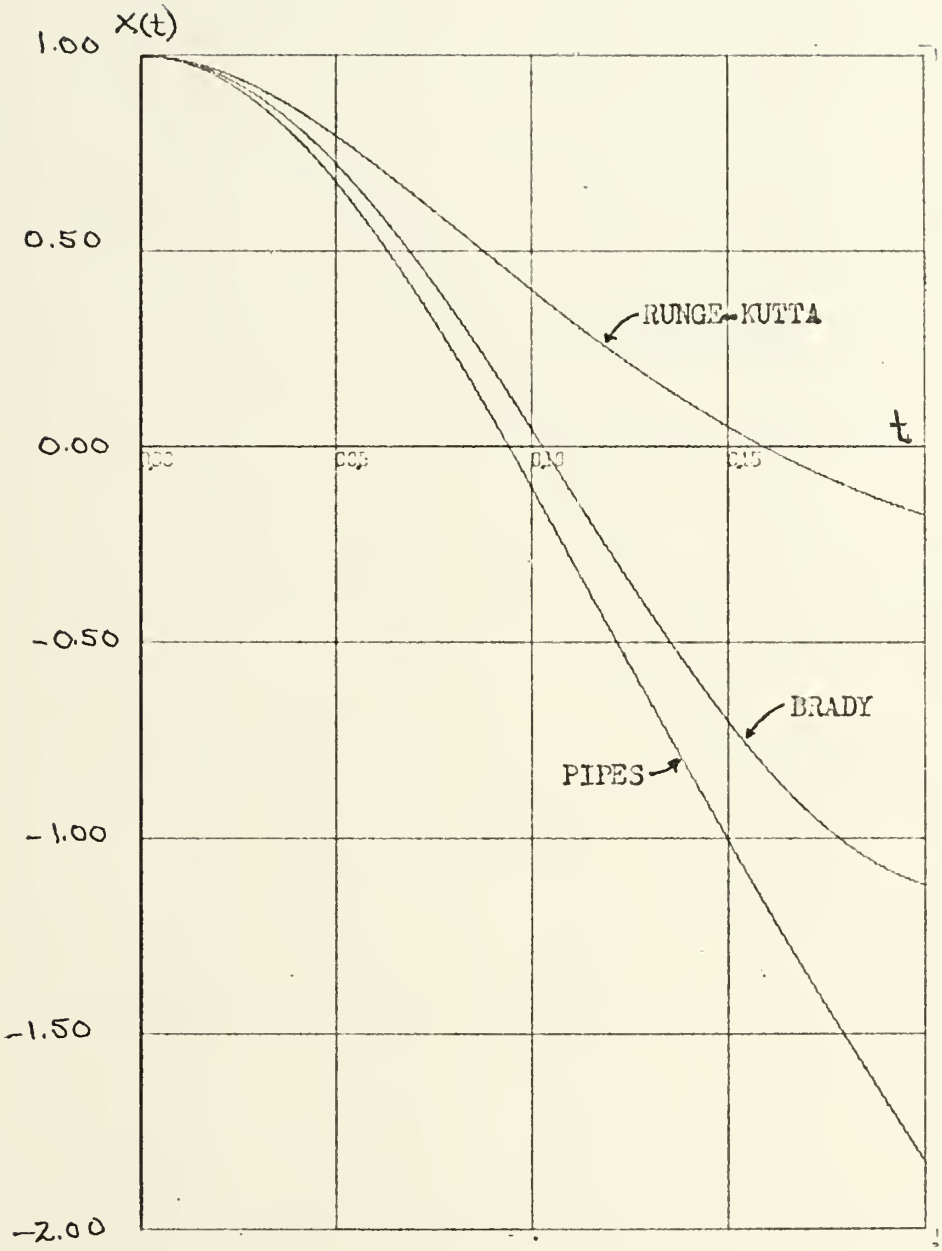

FIGURE 8. Time solution to $\ddot{x} + x + x + x^2 = 0$, $x_o = 1.0$

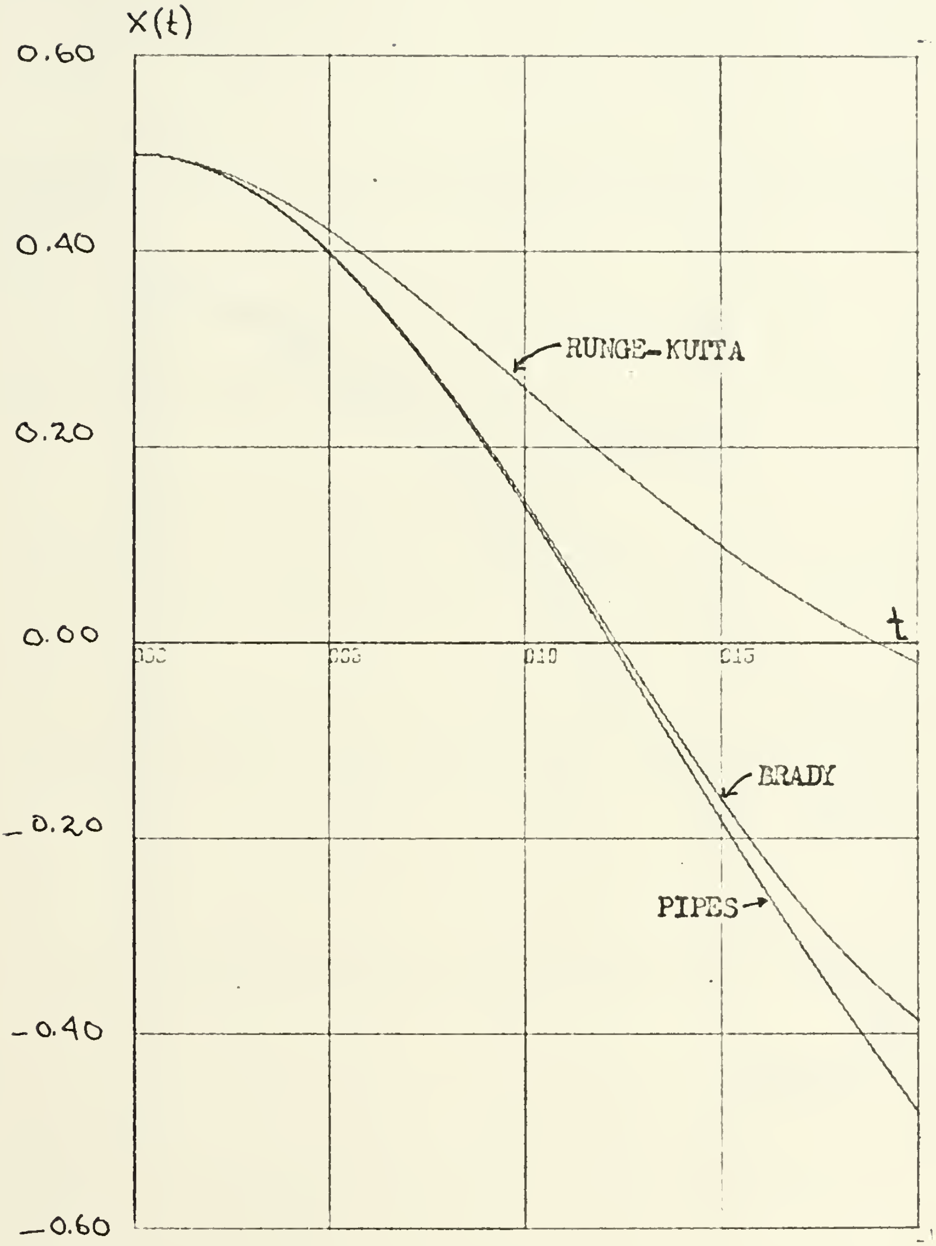

FIGURE 9. Time solution of $\ddot{x} + x + x + x^2 = 0$, $x_o = 0.50$

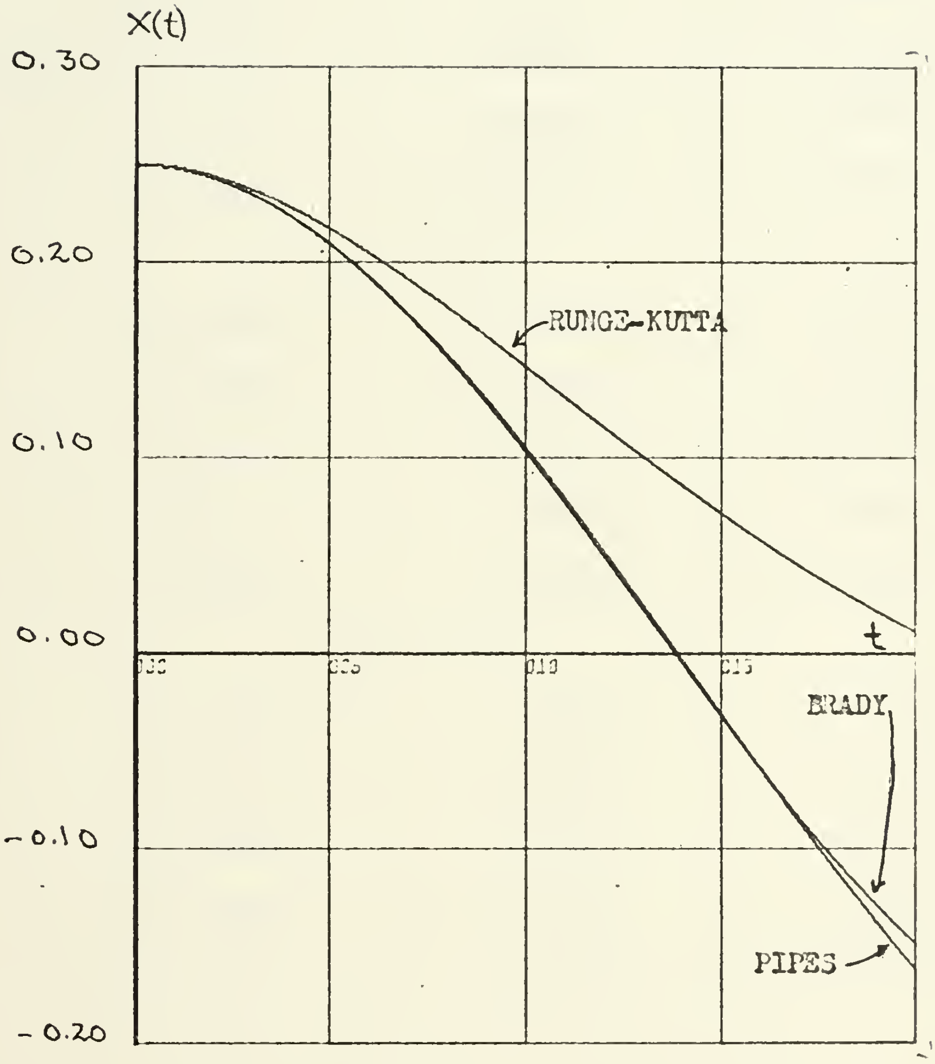

FIGURE 10. Time solution of $\ddot{x} + x + x + x^2 = 0$, $x_o = 0.25$

TABLE II

$x_o = 1.0$

Time	Runge-Kutta	Brady	Pipes
0.000	1.00000	1.00000	1.00000
0.050	0.99754	0.99719	0.99667
0.100	0.99035	0.98877	0.98669
0.150	0.97870	0.97478	0.97014
0.200	0.96290	0.95530	0.94711
		$x_o = 0.50$	
0.000	0.50000	0.50000	0.50000
0.050	0.49908	0.49898	0.49896
0.100	0.49638	0.49594	0.49584
0.150	0.49200	0.49089	0.49066
0.200	0.48604	0.48384	0.48344
		$x_o = 0.25$	
0.000	0.25000	0.25000	0.25000
0.050	0.24962	0.24959	0.24960
0.100	0.28849	0.24836	0.24839
0.150	0.24666	0.24632	0.24638
0.200	0.24417	0.24347	0.24357

$$\left.\begin{array}{l} x(0) = x_0 \\ \dot{x}(0) = 0 \end{array}\right\} \qquad (3.3.3)$$

1. Solution by Pipes Technique

Using operator notation, the a_i coefficients are identified as

$$\left.\begin{array}{l} a_1 = D^2 + 1 \\ a_2 = 0 \\ a_3 = 1 \\ k = 1 \\ \phi(t) = 0 \end{array}\right\} \qquad (3.3.4)$$

The $A_i(t)$ coefficients are found next. From equation (2.1.7)

$$A_1(t) = \phi(t)/a_1 \qquad (2.1.7)$$

or

$$(D^2+1)A_1(t) = 0 \qquad (3.3.5)$$

Taking Laplace transforms and impressing the initial conditions on $A_1(t)$ the solution is immediately seen to be the same as that of the quadratic case

$$A_1(t) = x_0 \cos t \qquad (3.3.6)$$

Substitution of a_2 into equation (2.1.4), gives coefficient A_2 as zero. A_3 is found from equation (2.1.4):

$$A_3(t) = -\frac{1}{a_1}\left[2a_2 A_1(t) A_2(t) + a_3 A_1^3(t)\right] \qquad (2.1.8)$$

$$A_3(t) = -\frac{1}{D^2+1}\left[0 + x_0^3 \cos^3 t\right] \qquad (3.3.7)$$

and after substituting for $\cos^3 t$, equation (3.3.7) becomes

$$A_3(t) = \frac{-x_0^3}{4}\left[\frac{3\cos t + \cos 3t}{D^2+1}\right] \qquad (3.3.8)$$

Taking Laplace transforms of both sides

$$A_3(s) = -\frac{x_0^3}{4}\left[\frac{3s}{(s^2+1)^2} + \frac{s}{(s^2+9)(s^2+1)}\right] \qquad (3.3.9)$$

The inverse transforms for the factors of equation (3.3.9) are found in reference 11, and the solution of $A_3(t)$ is

$$A_3(t) = -\frac{x_0^3}{8}\left[3t\sin t + \frac{1}{4}\cos t - \frac{1}{4}\cos 3t\right] \qquad (3.3.10)$$

Since all even a_i and thus even A_i are zero, A_4 is zero. The coefficient A_5, after substitution, reduces to

$$A_5(t) = -\frac{1}{a_1}\left[3a_3 A_1^2(t) A_3(t)\right] \qquad (3.3.11)$$

Substituting for the coefficients and manipulating trigonometric identities,

$$A_5(t) = \frac{3x_0^5}{32(D^2+1)}\left[3t\sin 3t + 3t\sin t + \tfrac{1}{2}\cos t - \tfrac{1}{4}\cos 3t - \tfrac{1}{4}\cos 5t\right] \qquad (3.3.12)$$

and taking Laplace transforms

$$A_5(s) = \frac{3x_0^5}{32}\left[\frac{18s}{(s^2+9)^2(s^2+1)} + \frac{6s}{(s^2+1)^3} + \frac{\frac{1}{2}s}{(s^2+1)^2} - \frac{\frac{1}{4}s}{(s^2+9)(s^2+1)} - \frac{\frac{1}{4}s}{(s^2+25)(s^2+1)}\right] \qquad (3.3.13)$$

The last three terms have readily available inverse transforms; however, the first and second terms require special techniques.

First term is evaluated by convolution:

Let

$$Z_1(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right] = \cos 3t \qquad (3.3.13a)$$

$$Z_2(t) = \mathcal{L}^{-1}\left[\frac{1}{(s^2+1)(s^2+9)}\right] = \frac{3t\sin t - \sin 3t}{24} \qquad (3.3.13b)$$

then

$$18[Z_1(t) * Z_2(t)] = \frac{3}{4}\int_0^t \cos 3\tau \left[3\cos(t-\tau) - \sin 3(t-\tau)\right] dt \tag{3.3.13c}$$

$$= \frac{9}{32}\cos t - \frac{25}{96}\cos 3t - \frac{3}{24}t\sin t \tag{3.3.13d}$$

The evaluation of the second term is found in Appendix A:

$$\mathcal{L}^{-1}\left[\frac{6s}{(s^2+1)^3}\right] = \frac{3}{4}\left[t\sin t - t^2\cos t\right] \tag{3.3.13e}$$

The remaining terms of (3.3.13) are found in reference 10.

The result for $A_5(t)$ becomes

$$A_5(t) = \frac{3x_0^5}{32}\left[\frac{1}{96}\left\{23\cos t - 24\cos 3t + \cos 5t\right\} + t\left(\sin t - \frac{3}{24}\sin 3t\right) - \frac{3}{4}t^2\cos t\right] \tag{3.3.14}$$

The approximate solution by Pipes method is the truncated series

$$x(t) \approx x_0\cos t - \frac{x_0^3}{8}\left[3t\sin t + \frac{1}{4}\cos t - \frac{1}{4}\cos 3t\right] + \frac{3x_0^5}{32}\left[\frac{1}{96}\left\{23\cos t - 24\cos 3t + \cos 5t\right\} + t\left(\sin t - \frac{3}{24}\sin 3t - \frac{3}{4}t^2\cos t\right] + \cdots \tag{3.3.15}$$

It is seen from equation (3.3.15) that the first secular term occurs in the second approximation.

2. Solution by Brady-Baycura Method

Upon taking Laplace transforms and using the nonlinear expression of equation (2.2.2) and impressing the initial conditions, equation (3.3.2) becomes

$$s^2 X^3(s) + (s^2+1)X(s) = s x_0 \tag{3.3.16}$$

Considering equation (3.3.16) to be a series, the method of reversion outlined in Chapter III, Section B, paragraph 2 can be used here also to obtain all the terms:

$$B_1 = \frac{1}{b_1} = \frac{1}{s^2+1} \tag{3.3.17a}$$

$$B_2 = 0 \tag{3.3.17b}$$

$$B_3 = \frac{1}{b_1^5}\left(2b_2^2 - b_1 b_3\right) = -\frac{b_3}{b_1^4} = -\frac{s^2}{(s^2+1)^4} \tag{3.3.17c}$$

The solution in terms of X(s) is

$$X(s) \approx \frac{x_0 s}{s^2+1} - \frac{x_0^3 s^5}{(s^2+1)^4} + \frac{3x_0^5}{(s^2+1)^7} - \cdots \tag{3.3.18}$$

The inverse transform of the first term is just x_0 cos t. The second term must be solved using the technique outlined in Brady (Ref. 2) and in the example of Chapter III, Section A, paragraph 2. The method is straightforward, giving the second term as

$$-\frac{x_0^3}{48}\left[15t\sin t - t^3\sin t + 9t^2\cos t\right] \qquad (3.3.19)$$

The second term (3.3.19) yields the higher order secular terms t^2 cos t and t^3 sin t which will be small contributions for $t \angle 1$.

It is also known from equation (3.1.19) that

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^7}\right] = \frac{\pi^{1/2}\, t^{13/2}\, J_{11/2}(t)}{2^{13/2}\,\Gamma(7)\cdot 1} \qquad (3.3.20)$$

By observing the result in second term, one sees that even higher order (t^6) secular terms will result from the third term of equation (3.3.18), and further that the coefficient for this third term is small (less than $.2 \times 10^{-4}$); therefore, little is gained by solving equation (3.3.20).

The approximate solution by the Brady-Baycura technique is

$$X(t) \approx x_0\cos t - \frac{x_0^3}{48}\left[48t\sin t - t^3\sin t + 9t^2\cos t\right] + \ldots \qquad (3.3.21)$$

3. Comparison of Solutions

Figures 11 and 12 show the phase plane solution for equation (3.3.2). Here the solution is seen to be stable for all x_o. The system always remains in a limit cycle.

It is also observed from Figures 13, 14 and 15 that for larger initial conditions, the Brady-Baycura method gives a better approximation to the solution for early time intervals. For the case of x_o small, Pipe's method gives better results. This is caused by the location and number of secular terms appearing in the solution, which causes it to diverge.

Selected values for the graphs of Figures 13, 14 and 15 are given in Table III.

D. NONLINEAR SPRING WITH DRIVING FUNCTION

Consider the system of Chapter III, Section A, but now as a forced system. The system nonlinear differential equation is

$$\frac{md^2x}{dt^2} + F(x) = B\cos\omega_o t \qquad (3.4.1)$$

where

$$F(x) = kx + bx^2 \qquad (3.4.2)$$

Letting all coefficients be unity, equation (3.4.1) becomes

$$\frac{d^2x}{dt^2} + x + x^2 = B\cos\omega_o t \qquad (3.4.3)$$

$$\left.\begin{aligned} x(0) &= x_o \\ \dot{x}(0) &= 0 \end{aligned}\right\} \qquad (3.4.4)$$

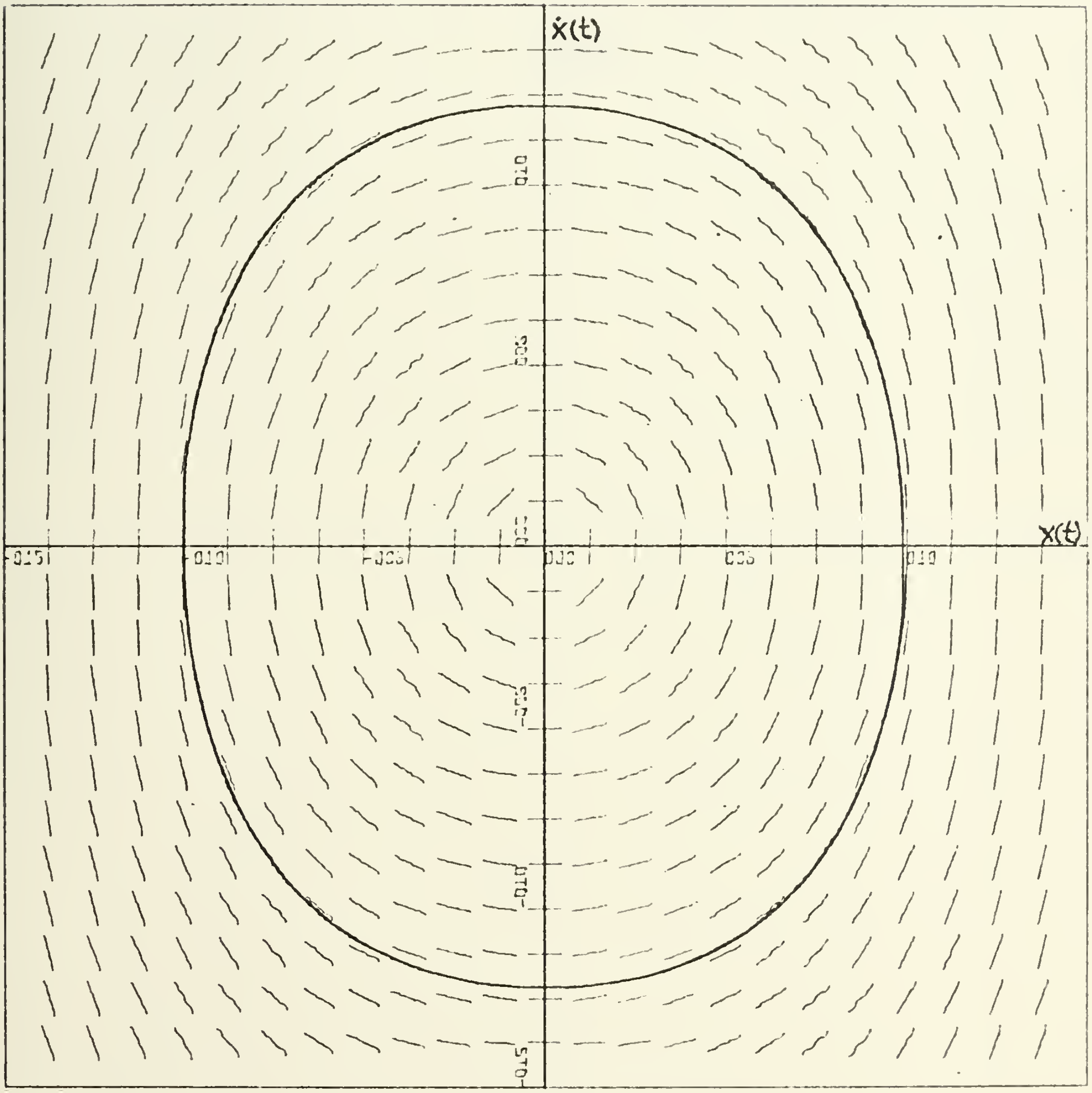

FIGURE 11. Phase portrait of $\ddot{x} + x + x^3 = 0$, $x_o = 1.0$

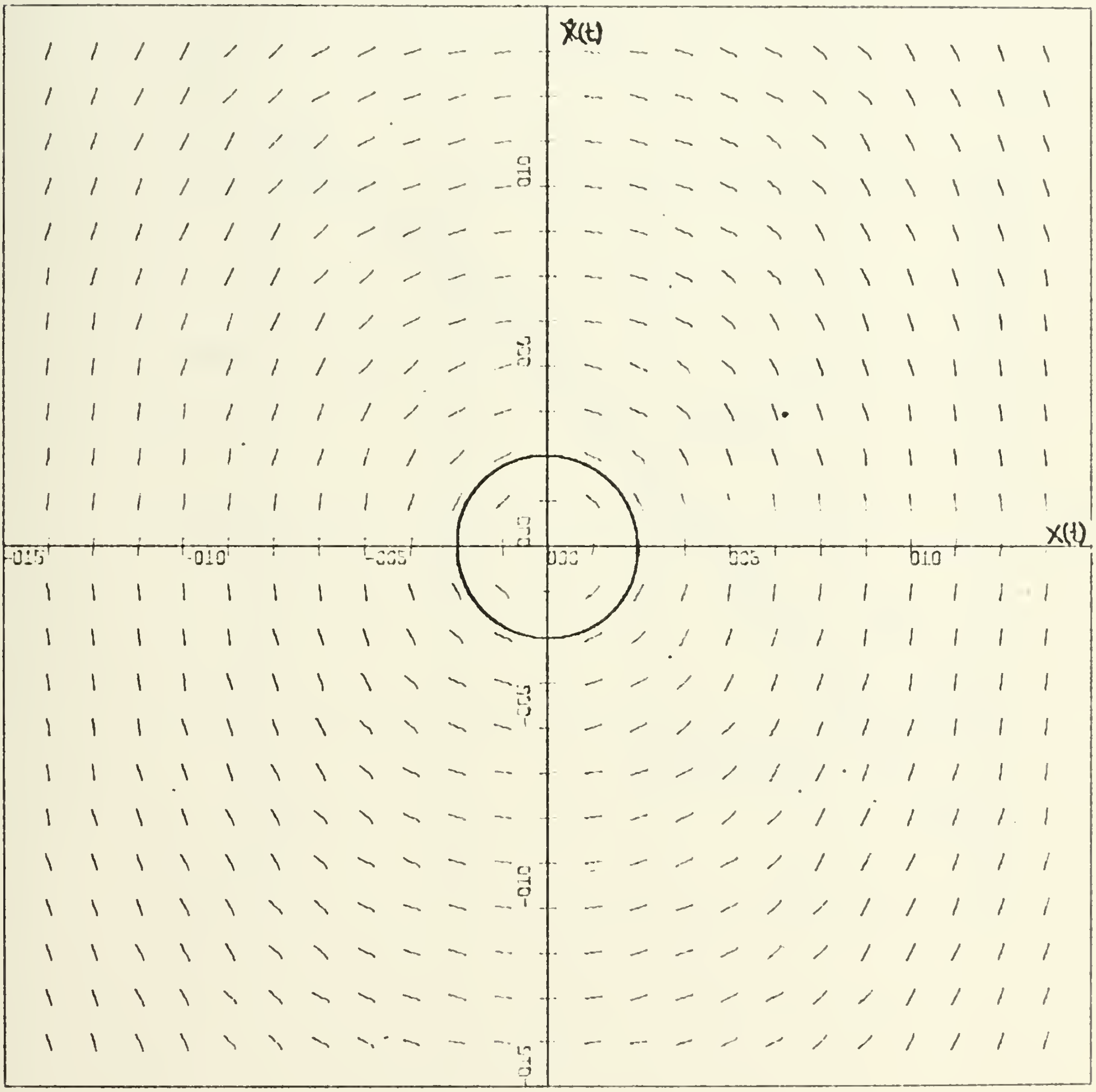

FIGURE 12. Phase portrait of $\ddot{x} + x + x^3 = 0$, $x_o = 0.25$

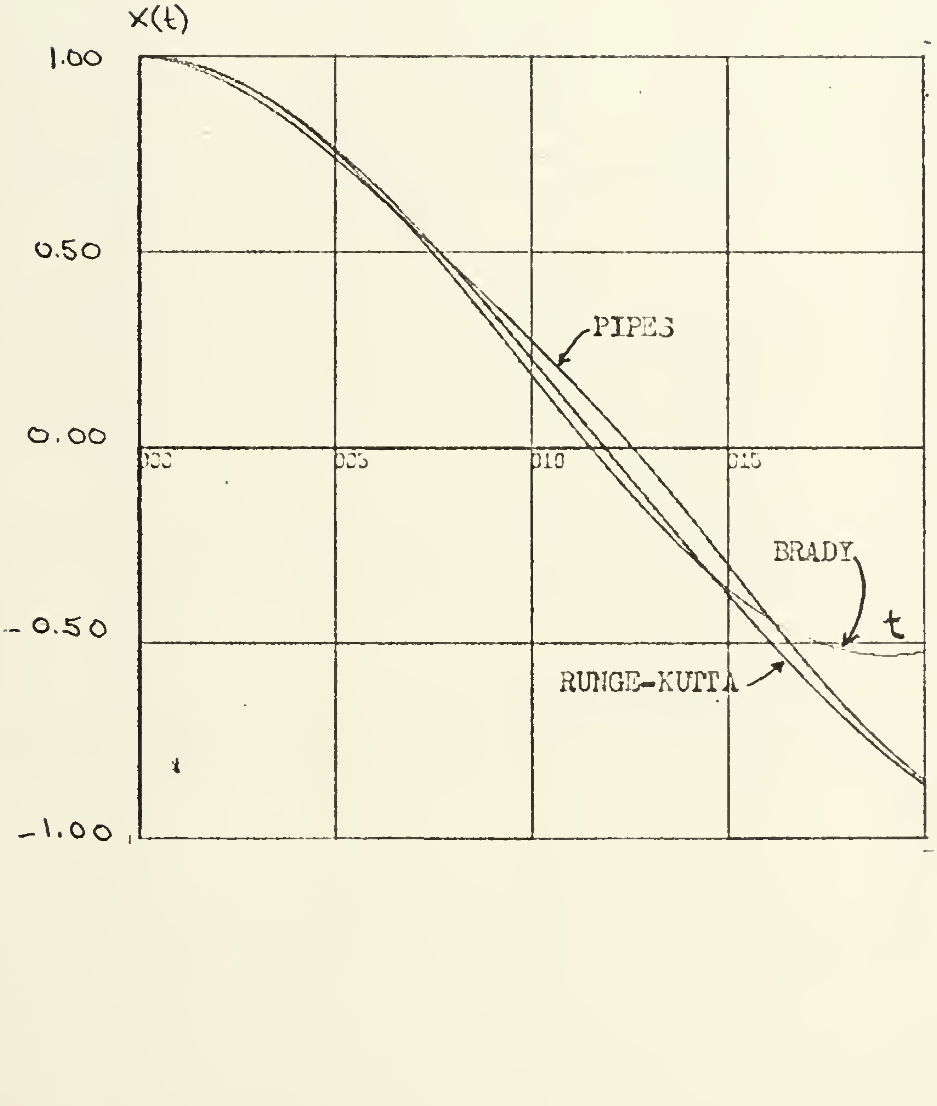

FIGURE 13. Time solution of $\ddot{x} + x + x^3 = 0$, $x_o = 1.0$

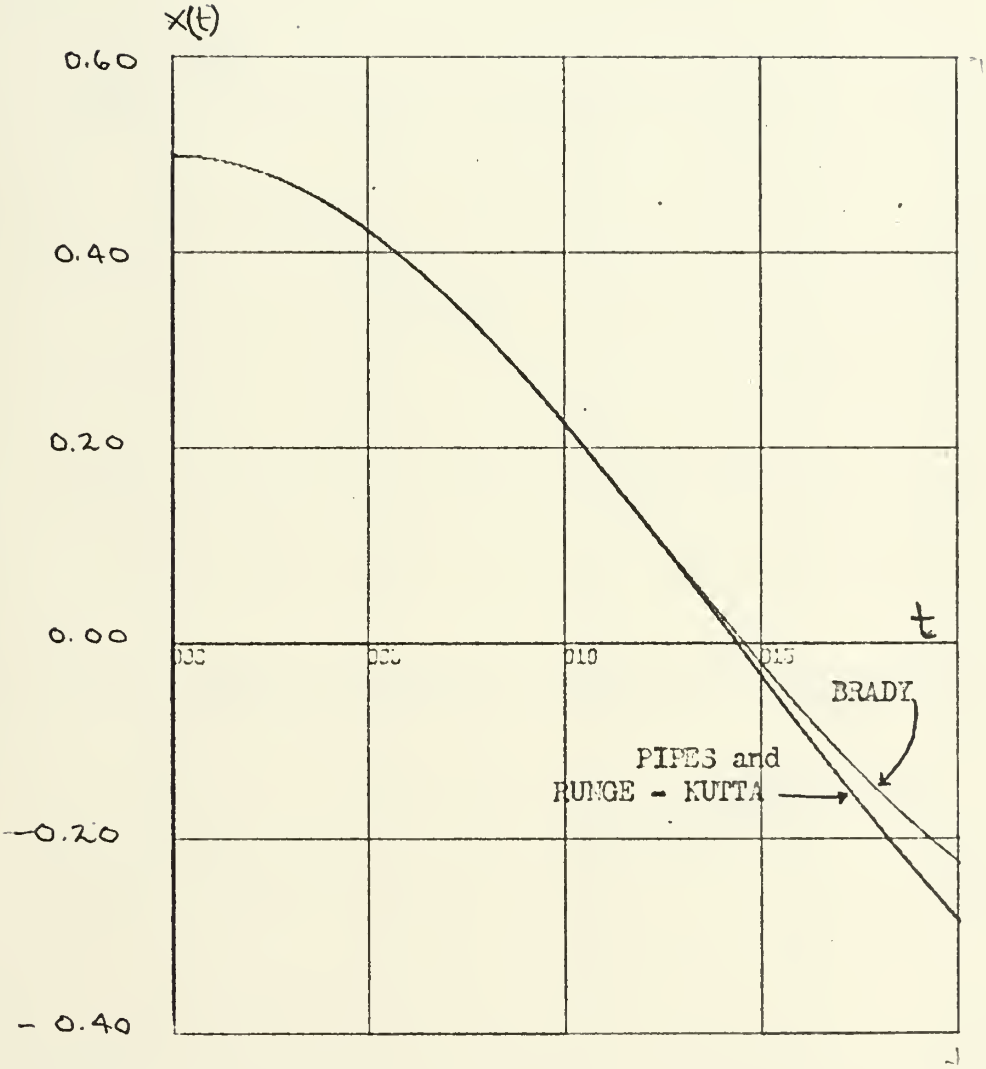

FIGURE 14. Time solution of $\ddot{x} + x + x^3 = 0$, $x_o = 0.50$

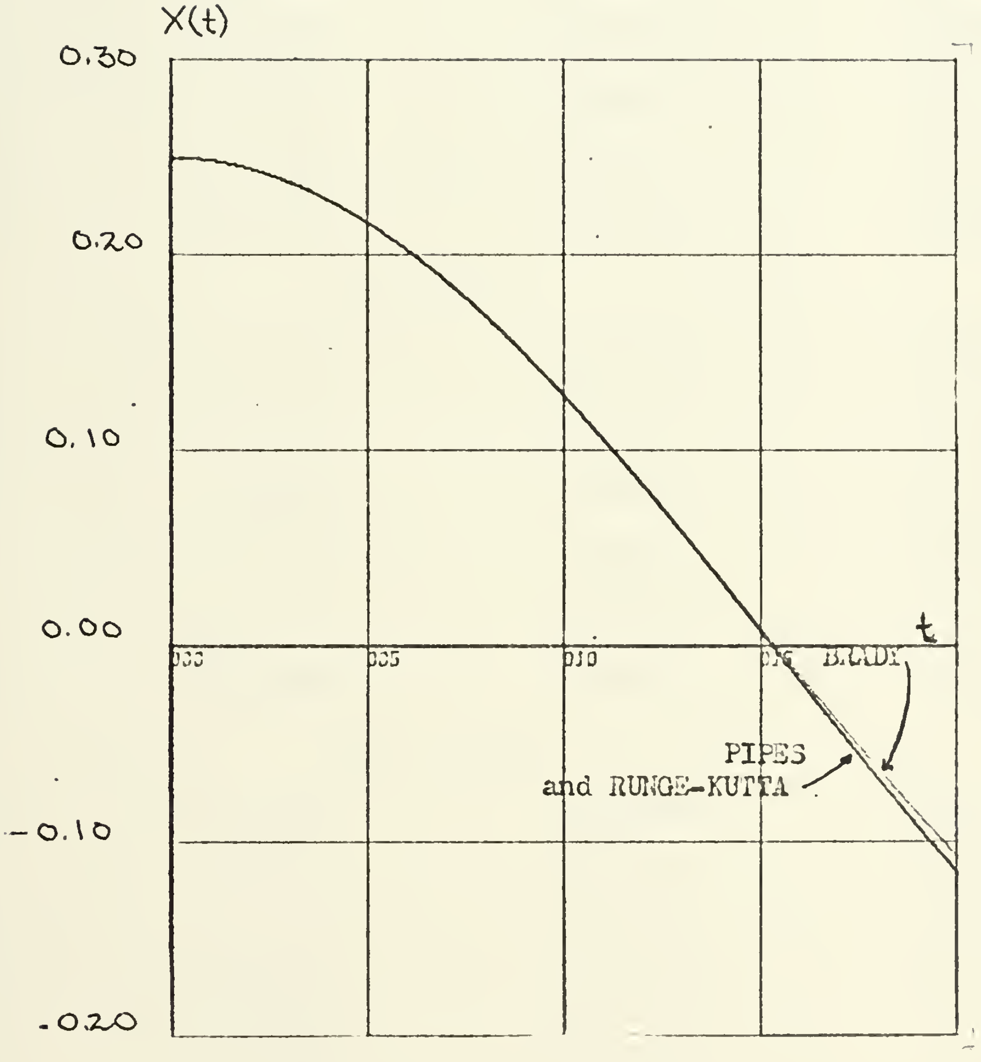

FIGURE 15. Time solution of $\ddot{x} + x + x^3 = 0$, $x_o = 0.25$

TABLE III

$x_o = 1.0$

Time	Runge-Kutta	Pipes	Brady
0.000	1.00000	1.00000	1.00000
0.050	0.99750	0.99768	0.99750
0.100	0.99003	0.99073	0.99002
0.150	0.97767	0.97920	0.97761
0.250	0.93877	0.94277	0.93831
0.500	0.76880	0.78095	0.76278
1.000	0.23370	0.25179	0.19357
	$x_o = 0.5$		
0.000	0.50000	0.50000	0.50000
0.050	0.49922	0.49922	0.49922
0.100	0.49688	0.49690	0.49688
0.250	0.48064	0.48077	0.48063
0.500	0.42461	0.42498	0.42444
1.000	0.22682	0.22707	0.22681
	$x_o = 0.25$		
0.000	0.25000	0.25000	0.25000
0.050	0.24967	0.24967	0.24967
0.300	0.23815	0.23816	0.23815
0.750	0.17928	0.17930	0.17931
1.000	0.12946	0.12947	0.12966
1.500	0.0089567	0.0089509	0.011029

1. Solution by Pipes Method

The a_i coefficients are those of equation (3.1.6), except here it is seen that $\emptyset(t) = B \cos \omega_o t$.

The solution proceeds as before

$$A_1(t) = \phi(t)/a_1 = \frac{B\cos\omega_0 t}{D^2+1} \tag{3.4.5}$$

Taking Laplace transforms and impressing initial conditions,

$$A_1(s) = \frac{sX_0}{s^2+1} + \frac{sB}{(s^2+\omega_0^2)(s^2+1)} \tag{3.4.6}$$

The terms of $A_1(s)$ have transforms in Ref. 10.

$$A_1(t) = x_0 \cos t + \frac{B}{\omega_0^2-1}\left[\cos t - \cos\omega_0 t\right] \tag{3.4.7}$$

$$= k_1 \cos t - k_2 \cos\omega_0 t \tag{3.4.8}$$

where

$$\left.\begin{aligned} k_1 &= x_0 + \frac{B}{\omega_0^2-1} \\ k_2 &= \frac{B}{\omega_0^2-1} \end{aligned}\right\} \tag{3.4.8a}$$

The solutions to $A_2(t)$ and $A_3(t)$ are straighforward using the method outlined in Chapter II, Section A. Thus

$$A_2(t) = -\Big[k_3 - k_4 \cos t - \frac{k_1^2}{6}\cos 2t + k_5 \cos(\omega_0-1)t + k_6 \cos(\omega_0+1)t - k_7 \cos 2\omega_0 t\Big] \tag{3.4.9}$$

where

$$\left.\begin{aligned} k_3 &= \frac{k_1^2 + k_2^2}{2} \\ k_4 &= k_3 - \frac{k_1^2}{6} + \frac{k_1 k_2}{\omega_0}\left(\frac{1}{\omega_0+2} + \frac{1}{\omega_0-2}\right) - \frac{k_2^2}{4(4\omega_0^2-1)} \\ k_5 &= \frac{k_1 k_2}{\omega_0(\omega_0-2)} \\ k_6 &= \frac{k_1 k_2}{\omega_0(\omega_0+2)} \\ k_7 &= \frac{k_2^2}{2(4\omega_0^2-1)} \end{aligned}\right\} \qquad (3.4.9a)$$

The solution for $A_3(t)$ is given in Appendix B.

The complete solution by Pipes method is given by

$$x(t) = A_1(t) + A_2(t) + A_3(t) + \ldots \qquad (3.4.10)$$

2. Solution by Brady-Baycura Technique

Equation (3.4.3) can be transformed as in Chapter III, Section A, paragraph 2 using the nonlinear transform for $x^2(t)$. After impressing the initial conditions, the transformed system equation becomes

$$sX^2(s) + (s^2+1)X(s) = x_0 s + \frac{Bs}{s^2+\omega_0^2} \qquad (3.4.11)$$

Letting equation (3.4.11) be a series, then by the method of algebraic series reversion

$$X(s) = A_1\left(x_0 s + \frac{Bs}{s^2+\omega_0^2}\right) + A_2\left(x_0 s + \frac{Bs}{s^2+\omega_0^2}\right) + \ldots \qquad (3.4.12)$$

where the A_i coefficients are

$$\left.\begin{aligned} A_1 &= \frac{1}{s^2+1} \\ A_2 &= -\frac{s}{(s^2+1)^3} \\ A_3 &= \frac{2s^2}{(s^2+1)^5} \end{aligned}\right\} \qquad (3.4.13)$$

Thus

$$X(s) = \frac{1}{s^2+1}\left[sx_0 + \frac{Bs}{s^2+\omega_0^2}\right] - \frac{s}{(s^2+1)^3}\left[x_0 s + \frac{Bs}{s^2+\omega_0^2}\right]^2 + \cdots \qquad (3.4.14)$$

The first term of the series in equation (3.4.14) is easily evaluated by performing the algebra indicated and using the transform pairs of Ref. 10. Let the first term be $F_1(s)$, then

$$F_1(s) = \frac{x_0 s}{s^2+1} + \frac{Bs}{(s^2+1)(s^2+\omega_0^2)} \qquad (3.4.15)$$

and

$$f_1(t) = x_0\cos t + \frac{B}{\omega_0^2-1}\left[\cos t - \cos\omega_0 t\right] \qquad (3.4.16)$$

$$= k_1\cos t - k_2\cos\omega_0 t \qquad (3.4.17)$$

where

$$\left.\begin{aligned} k_1 &= x_0 + \frac{B}{\omega_0^2-1} \\ k_2 &= \frac{B}{\omega_0^2-1} \end{aligned}\right\} \qquad (3.1.18)$$

The result is not unlike the first approximation obtained in the previous method by Pipes.

Let the second term be $F_2(s)$, then

$$F_2(s) = -\frac{s}{(s^2+1)^3}\left[x_0 s + \frac{Bs}{s^2+\omega_0^2}\right]^2 \qquad (3.4.19)$$

$$= \left[\frac{x_0^2 s^3}{(s^2+1)^3} + \frac{2Bx_0 s^3}{(s^2+1)^3(s^2+\omega_0^2)} + \frac{B^2 s^3}{(s^2+1)^3(s^2+\omega_0^2)}\right] \qquad (3.4.20)$$

The terms of $F_2(s)$ are all uncommon transforms. The technique of finding the inverse transforms is as follows:

Since

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^3}\right] = \frac{1}{8}(t\sin t - t^2\cos t) \qquad (3.4.21)$$

As shown in Appendix A, the derivative method of finding inverse transforms is extended to equation (3.4.21).

$$\frac{d^2}{dt^2}\left[\tfrac{1}{8}(t\sin t - t^2\cos t)\right] = \frac{1}{8}\left[3t\sin t + t^2\cos t\right] \qquad (3.1.21a)$$

and the inverse transform of the first term of equation (3.4.20) is

$$\frac{x_0^2}{8}\left[3t\sin t + t^2\cos t\right] \tag{3.4.22}$$

The second term of $F_2(s)$ is evaluated by convolution. Except for a constant, it is the first term of $f_2(t)$ convolved with

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+\omega_0^2}\right]$$

or

$$2Bx_0\left[\tfrac{1}{8}(3t\sin t + t^2\cos t)\right] * \left[\frac{1}{\omega_0}\sin\omega_0 t\right] \tag{3.4.23}$$

The expression (3.4.23) can be evaluated as

$$\frac{Bx_0}{16\omega_0}\Big[k_{14}\cos t + k_{15}\cos\omega_0 t + k_{16}t\sin t + k_{17}t^2\cos t$$
$$- \frac{6t}{\omega_0-1}\sin(2\omega_0-1)t - \frac{6}{(\omega_0-1)^2}\cos(2\omega_0-1)t\Big] \tag{3.4.24}$$

where

$$\left.\begin{aligned}
k_{14} &= \frac{6}{(\omega_0+1)^2} - \frac{4}{(\omega_0+1)^3} - \frac{4}{(\omega_0-1)^3} \\
k_{15} &= \frac{6}{(\omega_0-1)^2} - \frac{6}{(\omega_0+1)^2} + \frac{4}{(\omega_0+1)^3} + \frac{4}{(\omega_0-1)^3} \\
k_{16} &= \frac{6}{\omega_0+1} - \frac{6}{\omega_0-1} - \frac{4}{(\omega_0+1)^2} + \frac{4}{(\omega_0-1)^2} \\
k_{17} &= \frac{2}{\omega_0-1} + \frac{2}{\omega_0+1}
\end{aligned}\right\} \tag{3.4.24a}$$

Except for a constant, the last term of $F_2(s)$ is simply the term of (3.4.24) convolved again with $\frac{1}{\omega_0}\sin\omega_0 t$. The result is found in Appendix C.

Clearly the method is too cumbersome to carry beyond this second approximation. The approximate solution by the Brady-Baycura technique is

$$x(t) \approx f_1(t) + f_2(t) + \ldots \tag{3.4.26}$$

3. Comparing the Solutions

The time plots of Figures 16, 17, 18 and 19 exhibit the relative accuracy of the approximations to the Runge-Kutta solution, for various combinations of x_o, B and ω_o.

From Table IV it is apparent that the approximations are good only for the first initial time units. Pipes method gives only a slightly better approximation than the Brady-Baycura technique. For the case $\omega_o = 5.0$, it is observed that the approximations by both methods improve. This is in keeping with the original restriction of the system to forcing functions far from resonance.

E. NONLINEAR SPRING WITH CUBIC TERM AND FORCING FUNCTION

Assume the describing equation to be

$$\frac{d^2x}{dt^2} + x + x^3 = B\cos\omega_0 t \tag{3.5.1}$$

with initial conditions

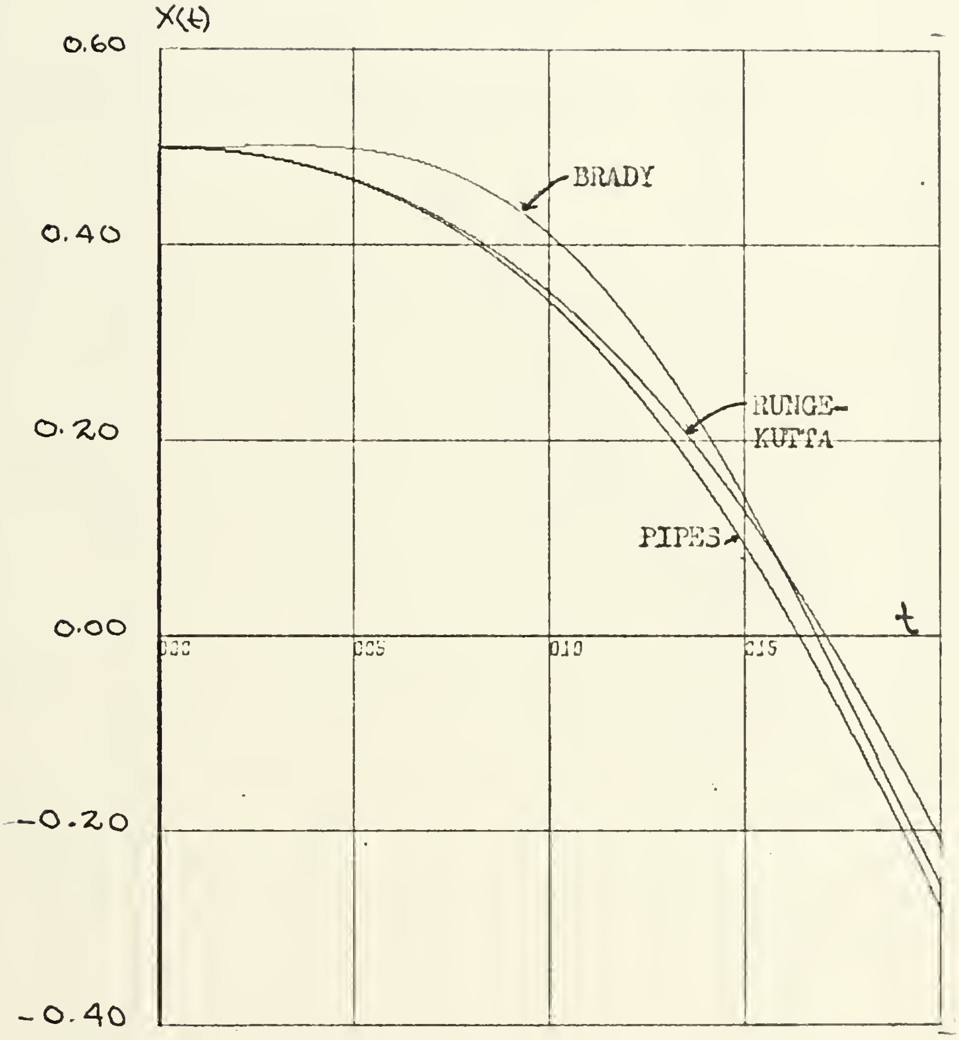

FIGURE 16. Time solution of $\ddot{x} + x + x^2 = 0.5 \cos 1.5t$, $x_o = 0.5$

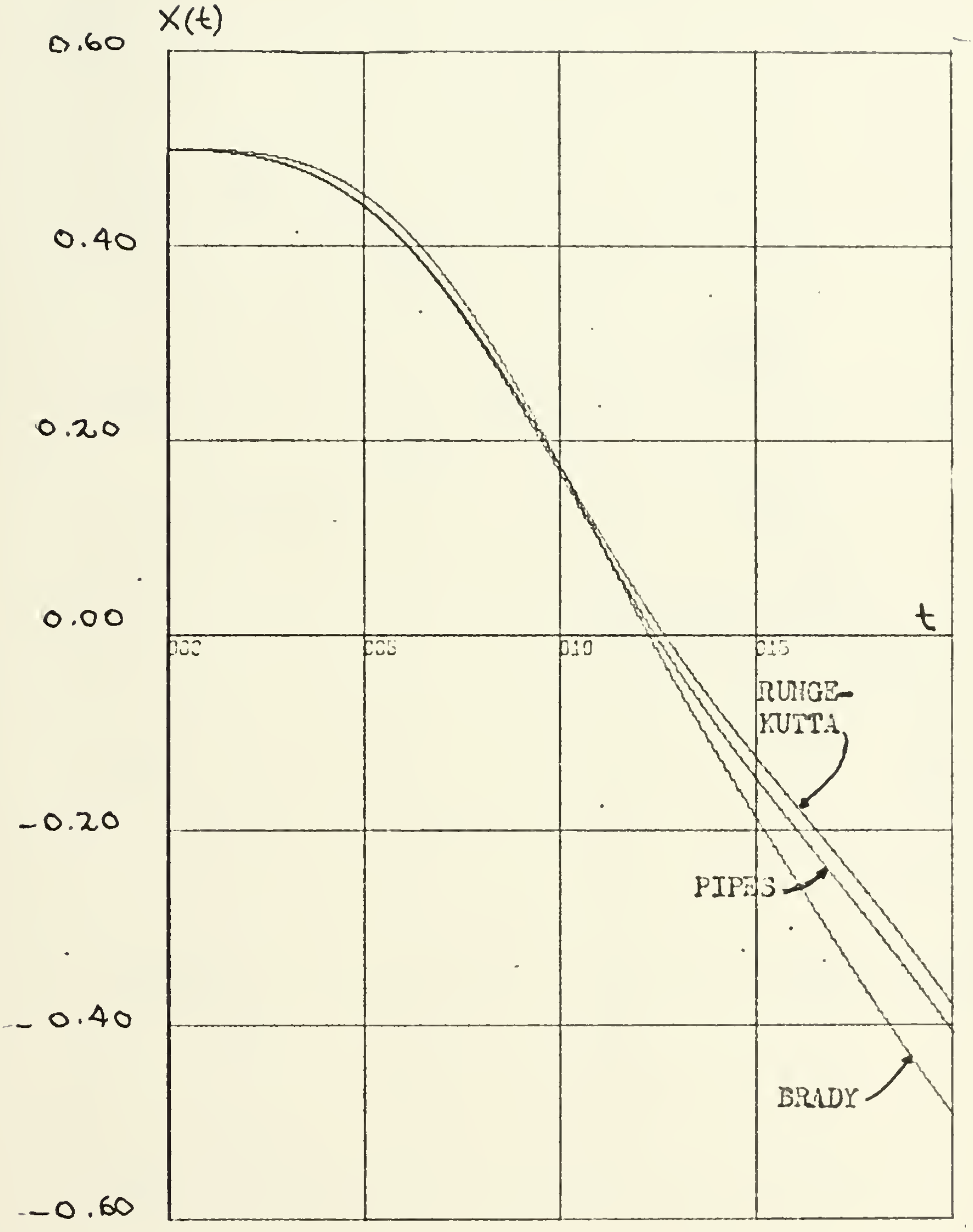

FIGURE 17. Time solution of $\ddot{x} + x + x^2 = 0.5 \cos 5.0t$, $x_o = 0.5$

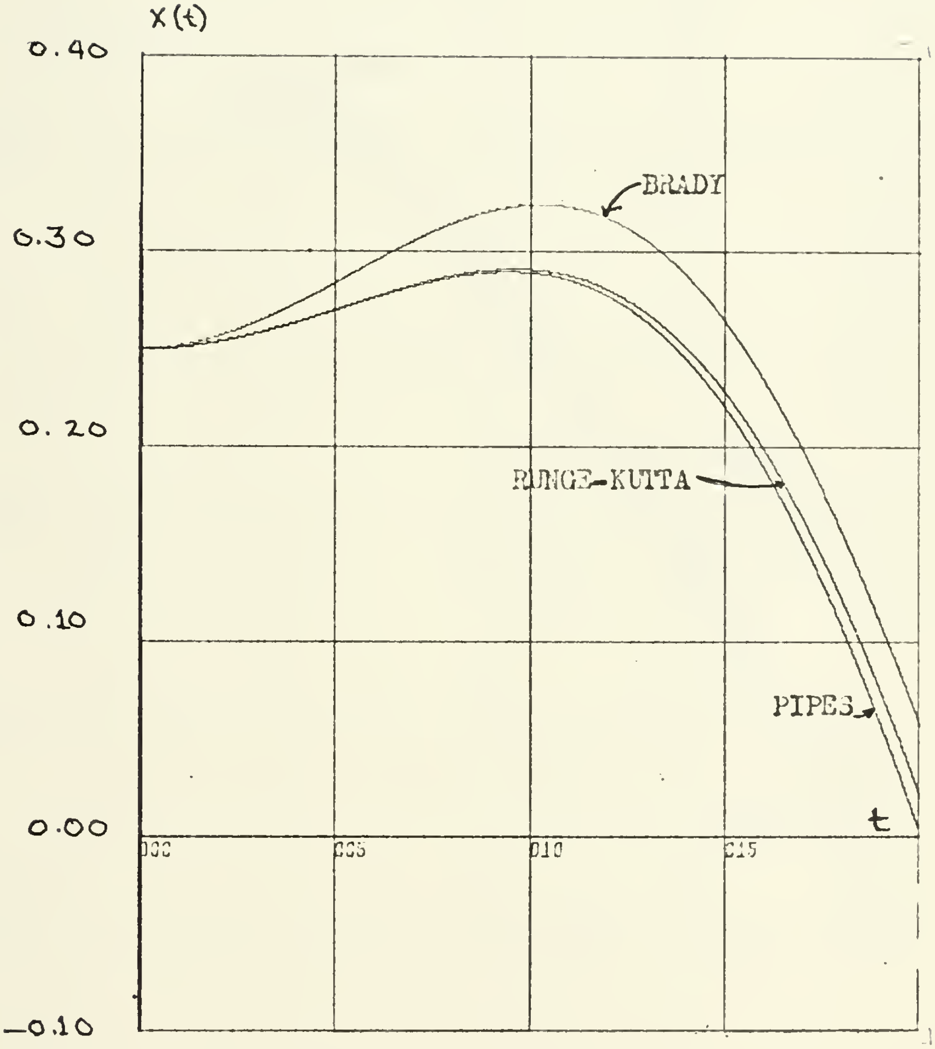

FIGURE 18. Solution to $\ddot{x} + x + x^2 = 0.5 \cos 1.5t$, $x_o = 0.25$

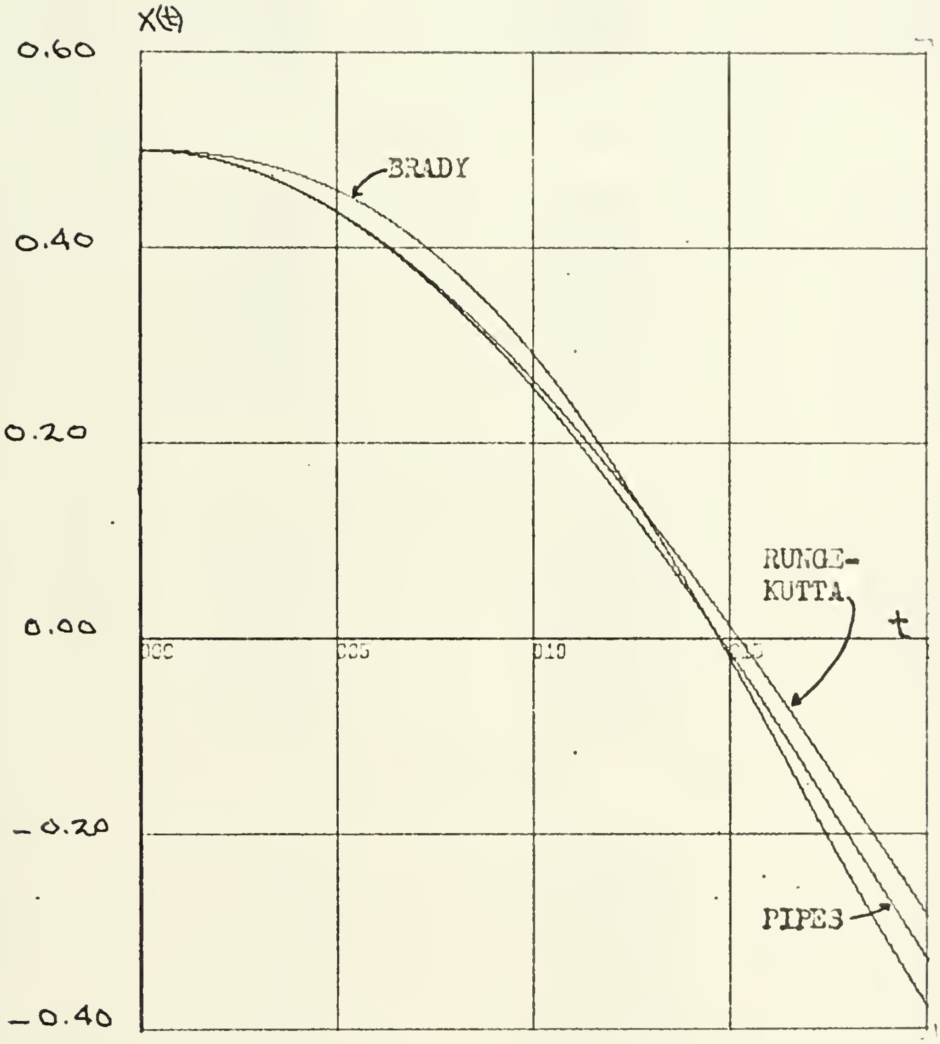

FIGURE 19. Time solution of $\ddot{x} + x + x^2 = 0.25 \cos 1.5t$, $x_o = 0.5$

TABLE IV

Time	Runge-Kutta	Pipes	Brady
$x_o = B = 0.5, \quad {}_o = 1.5$			
0.000	0.50000	0.50000	0.50000
0.050	0.49969	0.49969	0.50015
0.100	0.49875	0.49875	0.50053
0.250	0.49208	0.49205	0.50217
0.500	0.46719	0.46657	0.49946
1.000	0.35328	0.34448	0.41322
1.500	0.13099	0.09665	0.14546
$x_o = B = 0.5, \quad {}_o = 5.0$			
0.000	0.50000	0.50000	0.50000
0.050	0.49968	0.49968	0.49991
0.100	0.49870	0.49870	0.49955
0.250	0.49034	0.49030	0.49466
0.500	0.44401	0.44340	0.45529
1.000	0.17816	0.17054	0.17995
1.500	-0.12313	-0.14487	-0.18418
$x_o = 0.25, \ B = 0.5, \quad {}_o = 1.5$			
0.000	0.25000	0.25000	0.25000
0.050	0.25023	0.25023	0.25041
0.100	0.25093	0.25093	0.25161
0.250	0.25563	0.25563	0.25960
0.500	0.26987	0.26979	0.28343
1.000	0.29115	0.28972	0.32447
1.500	0.22963	0.22239	0.26635
$x_o = 0.5, \ B = 0.25, \quad {}_o = 1.5$			
0.000	0.50000	0.50000	0.50000
0.050	0.49937	0.49938	0.49972
0.100	0.49750	0.49750	0.49881
0.150	0.49438	0.49438	0.49722
0.250	0.48444	0.48441	0.49164
0.500	0.43861	0.43801	0.46008
1.000	0.26682	0.25904	0.29380
1.500	0.015343	-0.01124	-0.01725

$$\left.\begin{aligned} x(0) &= x_0 \\ \dot{x}(0) &= 0 \end{aligned}\right\} \qquad (3.5.1a)$$

1. Solution by Pipes Method

The solution proceeds in the usual way be identifying a_i's.

$$\left.\begin{aligned} a_1 &= D^2+1 \\ a_2 &= 0 \\ a_3 &= 1 \\ k &= 1 \\ \phi(t) &= B\cos\omega_0 t \end{aligned}\right\} \qquad (3.5.2)$$

Using equation (2.1.7), the first approximation A_1, is found by Laplace transform techniques.

$$A_1(t) = \frac{B\cos\omega_0 t}{D^2+1} \qquad (3.5.3)$$

$$A_1(s) = \frac{x_0 s}{s^2+1} + \frac{Bs}{(s^2+1)(s^2+\omega_0^2)} \qquad (3.5.4)$$

The solution is the same as for the quadratic case

$$A_1(t) = k_1\cos t - k_2\cos\omega_0 t \qquad (3.5.5)$$

where

$$\left.\begin{aligned} k_1 &= x_0 + \frac{B}{\omega_0^2-1} \\ k_2 &= \frac{B}{\omega_0^2-1} \end{aligned}\right\} \qquad (3.5.6)$$

The coefficient $A_2(t)$ was seen previously to be zero. Solving for $A_3(t)$ is straightforward and the result is

$$\begin{aligned} A_3(t) = -\Big[& \frac{k_3}{2} t \sin t - \Big(\frac{k_4}{\omega_0^2+1} + \frac{3k_1^2 k_4}{4[(\omega_0-2)^2-1]} + \frac{3k_1^2 k_2}{4[(\omega_0+2)^2-1]} \\ & - \frac{3k_1 k_2^2}{4[(2\omega_0+1)^2-1]} - \frac{3k_1 k_2^2}{4[(2\omega_0-1)^2-1]} + \frac{k_2^3}{4(9\omega_0^2-1)} \Big) \cos t + \frac{k_4}{\omega_0^2-1} \cos \omega_0 t \\ & + \frac{3k_1^2 k_4}{4[(\omega_0-2)^2-1]} \cos(\omega_0-2)t + \frac{3k_1^2 k_2}{4[(\omega_0+2)^2-1]} \cos(\omega_0+2)t \\ & - \frac{3k_1 k_2^2}{4[(2\omega_0+1)^2-1]} \cos(2\omega_0+1)t - \frac{3k_1 k_2^2}{4[(2\omega_0-1)^2-1]} \cos(2\omega_0-1)t \\ & + \frac{k_2^3}{4(9\omega_0^2-1)} \cos 3\omega_0 t \Big] \end{aligned} \tag{3.5.7}$$

where

$$\left.\begin{aligned} k_3 &= \frac{3k_1^3}{4} + \tfrac{3}{2} k_1 k_2^2 \\ k_4 &= \tfrac{3}{2} k_1^2 k_2 + \tfrac{3}{4} k_2^3 \end{aligned}\right\} \tag{3.5.8}$$

The solution of the undriven system of Chapter III, Section C should demonstrate the unfeasibility of carrying the approximation beyond $A_3(t)$. Furthermore, the first secular term in $A_3(t)$ will cause the solution to diverge.

2. Solution by Brady-Baycura Technique

Taking Laplace transforms of equation (3.5.1) and impressing the initial conditions, the system equation becomes

$$s^2 X^3(s) + (s^2+1)X(s) = sx_0 + \frac{Bs}{s^2+\omega_0^2} \tag{3.5.9}$$

Using the algebraic reversion of series technique to solve for X(s), let the left side of equation (3.5.9) be a series

$$y = f(s) = b_1 X(s) + b_2 X^2(s) + b_3 X^3(s) + \dots \tag{3.5.10}$$

then

$$X(s) = g(y) = B_1 y + B_2 y^2 + B_3 y^3 + \dots \tag{3.5.11}$$

$$= B_1 \left(s x_0 + \frac{Bs}{s^2 + \omega_0^2} \right) + B_2 \left(s x_0 + \frac{Bs}{s^2 + \omega_0^2} \right)^2 + \dots \tag{3.5.12}$$

Where from Ref. 9

$$B_1 = \frac{1}{b_1} = \frac{1}{s^2 + 1} \tag{3.5.13}$$

$$B_2 = -\frac{b_2}{b_1^3} = 0 \tag{3.5.14}$$

$$B_3 = \frac{1}{b_1^5} \left(2 b_2^2 - b_1 b_3 \right) = -\frac{b_3}{b_1^4} = \frac{-s^2}{(s^2 + 1)^4} \tag{3.5.15}$$

Then to first order approximation

$$X(s) = B_1 y \tag{3.5.16}$$

$$= \frac{s x_0}{s^2 + 1} + \frac{Bs}{(s^2 + \omega_0^2)(s^2 + 1)} \tag{3.5.17}$$

and

$$x(t) = k_1 \cos t - k_2 \cos \omega_0 t \tag{3.5.18}$$

where

$$\left.\begin{aligned} k_1 &= X_0 + \frac{B}{\omega_0^2 - 1} \\ k_2 &= \frac{B}{\omega_0^2 - 1} \end{aligned}\right\} \qquad (3.5.18a)$$

As expected, the result is equivalent to Pipes method for the first approximation.

Since B_2 is zero, the third term, $F_3(s)$, of the series is found

$$F_3(s) = -\frac{s^2}{(s^2+1)^4}\left[sX_0 + \frac{Bs}{s^2+\omega_0^2}\right]^3 \qquad (3.5.19)$$

$$= -\left[\frac{X_0^3 s^5}{(s^2+1)^4} + \frac{3BX_0^5 s^5}{(s^2+1)^4(s^2+\omega_0^2)} + \frac{3B^2X_0 s^5}{(s^2+\omega_0^2)(s^2+1)^4} + \frac{B^3 s^5}{(s^2+\omega_0^2)(s^2+1)^4}\right] \qquad (3.5.20)$$

The inverse transform of the first term as given in equation (3.3.18).

$$\mathcal{L}^{-1}\left[\frac{s^5}{(s^2+1)^4}\right] = \frac{1}{48}\left[15t\sin t + 9t^2\cos t - t^3\sin t\right] \qquad (3.5.21)$$

Except for the constants, the successive terms of $F_3(s)$ must be evaluated by successive convolution of $\frac{1}{\omega_0}\sin\omega_0 t$ with the right side of equation (3.5.21). While continuing the evaluation of $F_3(s)$ will lead to terms of cos t, modulation terms and higher resonance terms, it will also contribute numerous secular terms, and thus cause the solution to diverge more rapidly. Again the technique has lost its desirability by providing cumbersome transforms.

Using the first term only of $F_3(s)$, the solution is given by

$$x(t) \approx k_1 \cos t - k_2 \cos \omega_0 t - \frac{x_0^3}{48}\left[15 t \sin t + 9 t^2 \cos t - t^3 \sin t\right] + \dots \qquad (3.5.22)$$

3. Comparing the Solutions

Figures 20, 21, 22 and 23 are graphs of the approximate solutions and the true solution for various combinations of x_o, B and ω_o. Table V lists selected values from the solutions.

The graphs demonstrate that except in the extreme case of Figure 20 where $x_o = B$ and $\omega_o = 1.5$, that Pipes Method provides only a slightly better approximation than the Brady-Baycura Method. This is due to the number of terms generated by the Pipes Method. Figure 21 shows that the condition of the system operating far from resonance, Pipes Method gives a better solution for a longer period of time. Both approximations improve as x_o and B become small.

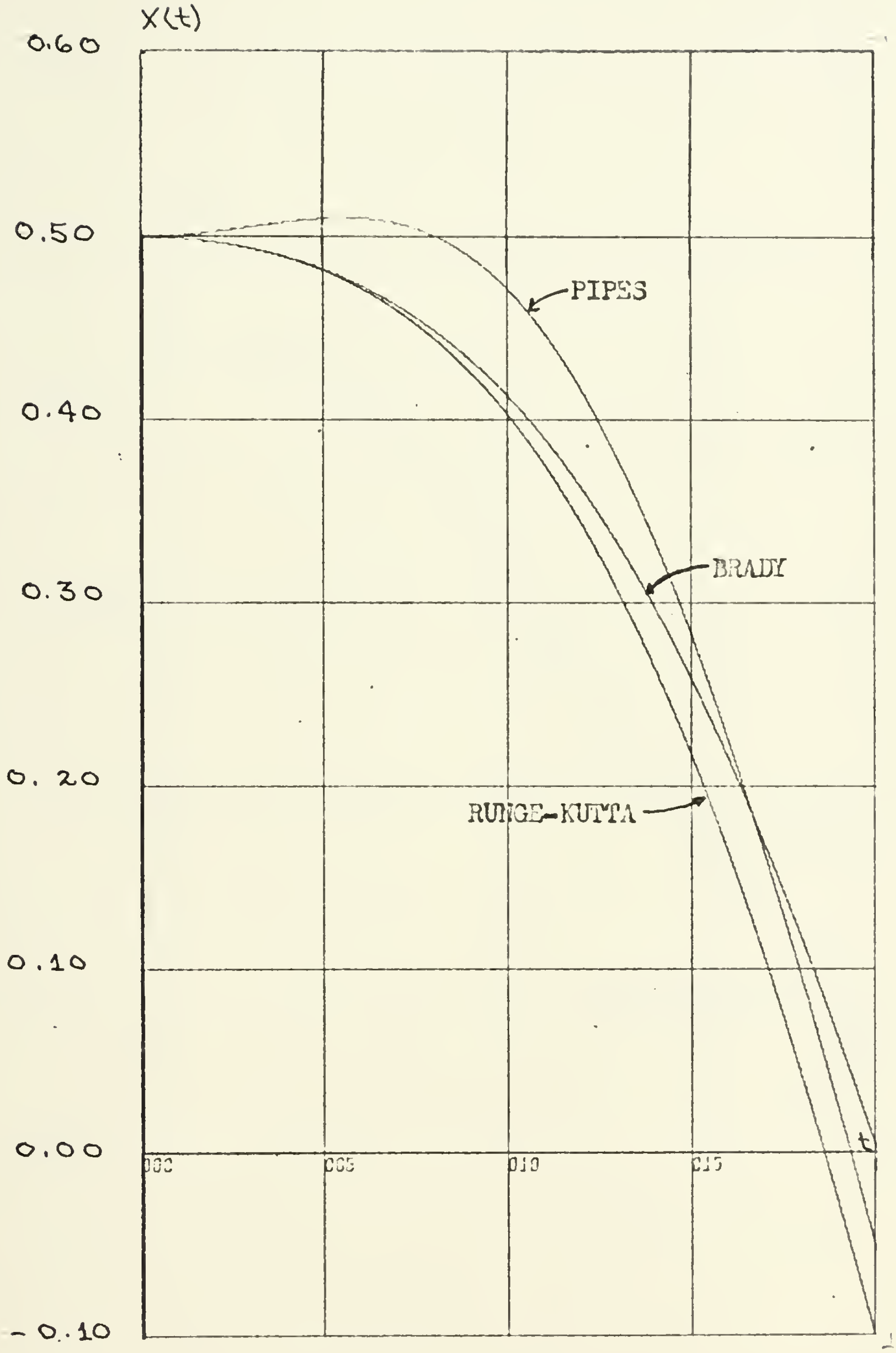

FIGURE 20. Time solution of $\ddot{x} + x + x^3 = 0.5 \cos 1.5t$, $x_o = 0.5$

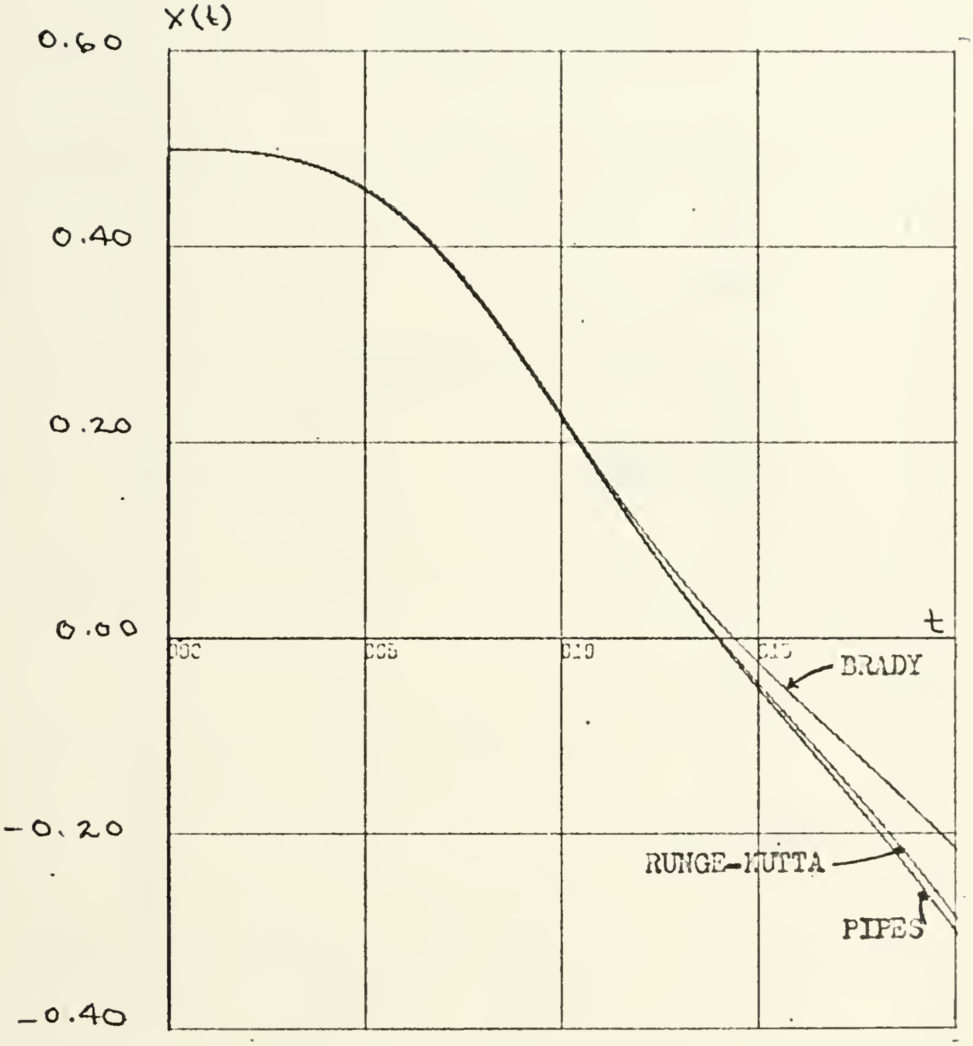

FIGURE 21. Time solution of $\ddot{x} + x + x^3 = 0.5 \cos 5.0t$, $x_o = 0.5$

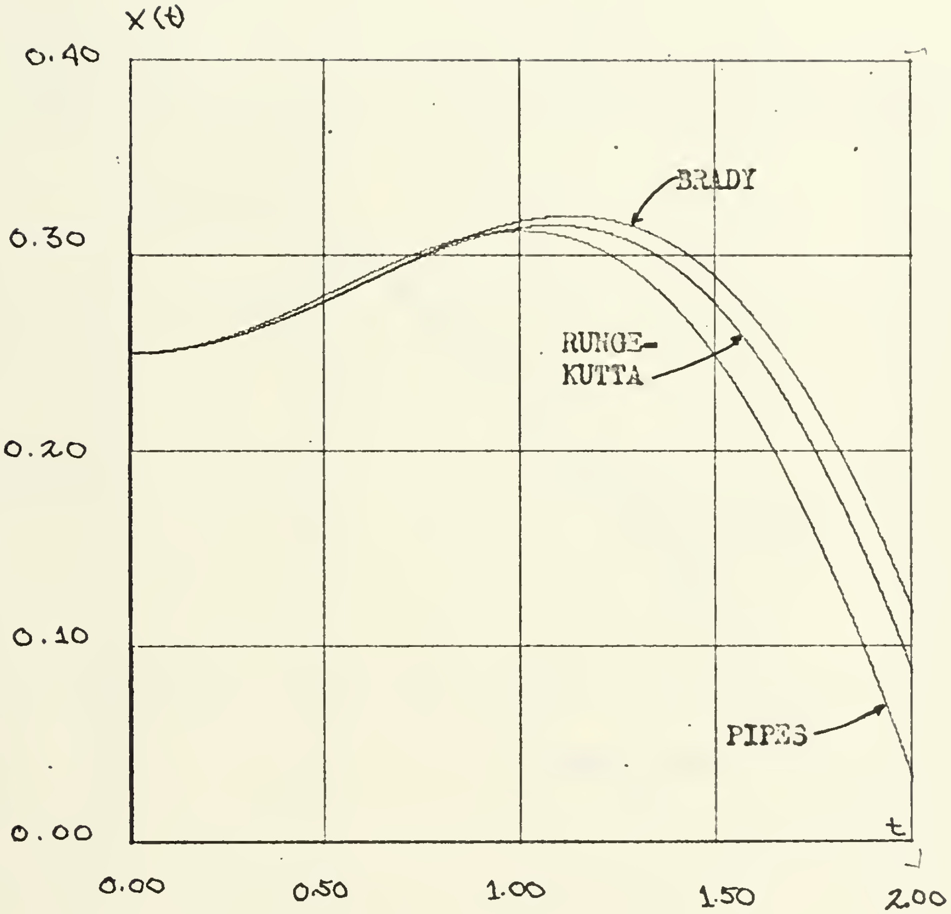

FIGURE 22. Time solution to $\ddot{x} + x + x^3 = 0.5 \cos 1.5t$, $x_o = 0.25$

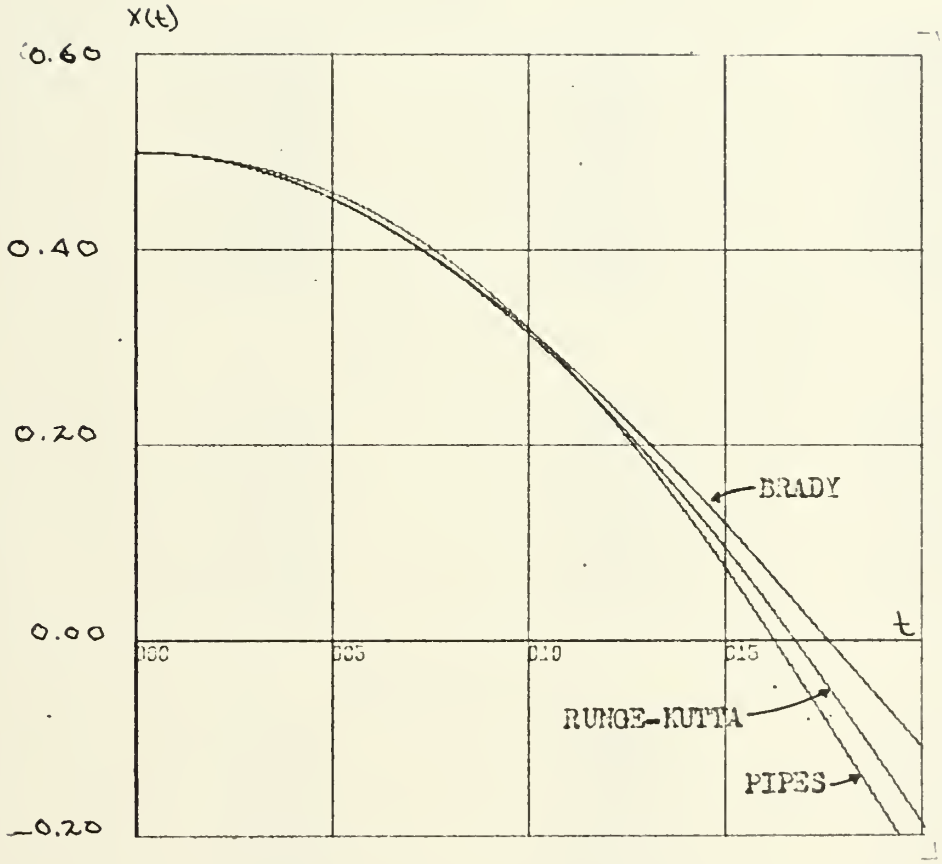

FIGURE 23. Time solution to $\ddot{x} + x + x^3 = 0.25 \cos 1.5t$, $x_o = 0.5$

TABLE V

Time	Runge-Kutta	Pipes	Brady
	$x_o = B = 0.5, \quad = 1.5$		
0.000	0.50000	0.50000	0.50000
0.050	0.49984	0.50017	0.49984
0.100	0.49937	0.50068	0.49937
0.150	0.49857	0.50150	0.49858
0.250	0.49594	0.50386	0.49599
0.500	0.48209	0.51016	0.48280
1.000	0.40483	0.47304	0.41464
1.500	0.22075	0.28681	0.26156
	$x_o = B = 0.50, \quad = 5.0$		
0.000	0.50000	0.50000	0.50000
0.050	0.49984	0.49988	0.49984
0.100	0.49932	0.49949	0.49933
0.150	0.49834	0.49870	0.49835
0.250	0.49420	0.49516	0.49425
0.500	0.45886	0.46195	0.45941
1.000	0.22743	0.23117	0.23216
1.500	-0.04477	-0.04985	-0.02377
	$x_o = 0.25, B = 0.50, \quad = 1.5$		
0.000	0.25000	0.25000	0.25000
0.050	0.25029	0.25034	0.25029
0.100	0.25116	0.25135	0.25117
0.250	0.25709	0.25816	0.25711
0.500	0.27572	0.27884	0.27596
1.000	0.31407	0.31302	0.31748
1.500	0.27665	0.25064	0.29059
	$x_o = 0.50, B = 0.25, \quad = 1.5$		
0.000	0.50000	0.50000	0.50000
0.050	0.49953	0.49962	0.49953
0.100	0.49812	0.49846	0.49813
0.150	0.49578	0.49653	0.49579
0.250	0.48829	0.49029	0.48831
0.500	0.45336	0.45969	0.45362
1.000	0.31626	0.32228	0.32072
1.500	0.09741	0.07727	0.12178

IV. CONCLUSIONS

It has been demonstrated that these various methods of obtaining approximate solutions to nonlinear systems with initial conditions are accurate only during the early time intervals. Further, it was shown that this degree of accuracy may be satisfactory for engineering applications.

It has also been shown that the Brady-Baycura technique of nonlinear transforms usually provides a more accurate solution; however, the ease and facility of Pipes Method might generally make this a more desirable method.

The method of handling so-called secular terms is still unresolved. It is suggested that they may be eliminated by letting the frequency also be a series such that the constants of the series are chosen to eliminate resonance conditions and thus eliminate the secular terms. The method would be similar to that given by Pipes in Ref. 12.

Both methods show that the higher order harmonic terms exist in the solution; however, the subharmonic terms fail to occur in the first few terms of the approximations.

Since the problem is an initial value problem, it appears that the solution could be improved by stopping the problem solution before it diverges from the true solution (t .5), then restarting with a new set of initial conditions. The effects of x(o) would have to be investigated. The value of such a method seems dubious since Runge-Kutta analysis on the digital computer seems an easier method.

APPENDIX A

INVERSE LAPLACE TRANSFORM OF $\frac{s}{(s^2+1)^3}$

The derivative technique is used:

From Ref. 10,

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right] = \frac{1}{2}\left[\sin t - t\cos t\right] \tag{A-1}$$

also

$$\frac{d}{ds}\left[\frac{1}{(s^2+1)^2}\right] = -\frac{4s}{(s^2+1)^3} \tag{A-2}$$

or

$$\frac{s}{(s^2+1)^3} = -\frac{1}{4}\frac{d}{ds}\left[\frac{1}{(s^2+1)^2}\right] \tag{A-3}$$

since

$$\mathcal{L}^{-1}\left[f^{(n)}(s)\right] = -1^n t^n F(t), \quad n = 1,2,3,\ldots \tag{A-4}$$

where n is the n^{th} derivative. Then combining equations (A-4) and (A-3)

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^3}\right] = -\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left[\frac{1}{(s^2+1)^2}\right]\right\} \tag{A-5}$$

$$= \frac{t}{8}\left[\sin t - t\cos t\right] \tag{A-6}$$

APPENDIX B

EVALUATION OF A_3 IN FORCED QUADRATIC SYSTEM

From equation (2.1.5) A_3 becomes

$$A_3(t) = -\frac{2a_2 A_1(t) A_2(t)}{a_1} \tag{B-1}$$

After substituting for a_1, a_2, A_1 and A_2 and combining terms,

$$\begin{aligned} A_3(t) = \frac{1}{D^2+1}\Big[& k_8 \cos t - k_1 k_4 \cos 2t - \frac{k_1^3}{6} \cos 3t - k_9 \cos \omega_0 t \\ & + k_{10} \cos(\omega_0 - 2)t + k_{11} \cos(\omega_0 + 2)t + k_2 k_4 \cos(\omega_0 + 1)t \\ & + k_2 k_4 \cos(\omega_0 - 1)t - k_{12} \cos(2\omega_0 + 1)t - k_{13} \cos(2\omega_0 - 1)t \\ & + k_2 k_7 \cos 3\omega_0 t - k_1 k_4 \Big] \end{aligned} \tag{B-2}$$

where

$$\begin{aligned} k_8 &= 2k_1 k_3 - \frac{k_1^3}{6} - k_2 k_5 - k_2 k_6 \\ k_9 &= 2k_1 k_3 - k_1 k_5 - k_2 k_7 - k_1 k_6 \\ k_{10} &= k_1 k_5 + \frac{k_1^2 k_2}{6} \\ k_{11} &= k_1 k_6 + \frac{k_1^2 k_2}{6} \\ k_{12} &= k_1 k_7 + k_2 k_6 \\ k_{13} &= k_2 k_5 + k_1 k_7 \end{aligned} \tag{B-3}$$

Except for the first term of A_3, the transforms are all of the form:

$$\frac{s}{(s^2+1)(s^2+\omega^2)} \tag{B-4}$$

For which the inverse transform is known:

$$\frac{1}{\omega^2-1}\left[\cos t - \cos \omega t\right] \tag{B-5}$$

The first term has the transform:

$$\frac{s}{(s^2+1)^2} \tag{B-6}$$

which has the inverse

$$\frac{t}{2} \sin t \tag{B-7}$$

Using expressions (B-5) and (B-7), A_3 becomes

$$\begin{aligned} A_3(t) = \Bigg[& \frac{k_8}{2} t \sin t - \left(\frac{k_1 k_4}{3} + \frac{k_1^3}{48} + \frac{k_9}{(\omega_0^2-1)} - \frac{k_{10}}{(\omega_0-2)^2-1} - \frac{k_{11}}{(\omega_0+2)^2-1} \right. \\ & \left. - \frac{k_2 k_4}{(\omega_0-1)^2-1} - \frac{k_2 k_4}{(\omega_0-1)^2-1} + \frac{k_{12}}{(2\omega_0+1)^2-1} + \frac{k_{13}}{(2\omega_0-1)^2-1} - \frac{k_2 k_7}{9\omega_0^2-1} \right) \cos t \\ & + \frac{k_1 k_4}{3} \cos 2t + \frac{k_1^3}{48} \cos 3t + \frac{k_9}{\omega_0^2-1} \cos \omega_0 t \\ & - \frac{k_{10}}{(\omega_0-2)^2-1} \cos(\omega_0-2)t - \frac{k_{11}}{(\omega_0+2)^2-1} \cos(\omega_0+2)t - \frac{k_2 k_4}{\omega(\omega+2)} \cos(\omega_0+1)t \\ & - \frac{k_2 k_4}{\omega_0(\omega_0-2)} \cos(\omega_0-1)t + \frac{k_{12}}{4\omega_0(\omega_0+1)} \cos(2\omega_0+1)t + \frac{k_{13}}{4\omega_0(\omega_0-1)} \cos(2\omega_0-1)t - \frac{k_2 k_7}{9\omega_0-1} \cos 3\omega_0 t \\ & - k_1 k_4 \Bigg] \end{aligned} \tag{B-8}$$

APPENDIX C

SOLUTION OF LAST TERM OF $F_2(s)$ BY BRADY-BAYCURA METHOD

Let the expression in parenthesis of (3.4.24) equal $Z_1(t)$ and $\frac{1}{\omega_o}$ sin $_o$ t equal $Z_2(t)$, then

$$Z_1(t) = f_1(t) + f_2(t) + \ldots \tag{C-1}$$

thus

$$\frac{B^2}{32\omega_o^2}\left[Z_1(t) * Z_2(t)\right] = f_1(t) * Z_2(t) + f_2(t) * Z_2(t) + \ldots \tag{C-2}$$

or six convolutions make up the last term. Since each of these convolutions is of the type previously evaluated, the solution is now straightforward and the last term becomes

$$\frac{B^2}{32\omega_o^2}\Bigl\{\frac{k_{14}}{2}\Bigl[\frac{2\omega_o}{\omega_o^2-1}\cos t + \frac{1}{\omega_o-1}\cos(\omega_o-1)t - \frac{2\omega_o}{\omega_o^2-1}\cos\omega_o t\Bigr]$$

$$+\frac{k_{15}}{8\omega_o}\Bigl[t(4\sin\omega_o t + \cos\omega_o t - \cos 3\omega_o t) - \cos\omega_o t + \cos 3\omega_o t\Bigr]$$

$$+\frac{k_{16}}{2}\Bigl[\Bigl(\frac{1}{(\omega_o+1)^2} - \frac{1}{(\omega_o-1)^2}\Bigr)\cos t + \Bigl(\frac{1}{(\omega_o-1)^2} - \frac{1}{(\omega_o+1)^2}\Bigr)\cos\omega_o t\Bigr]$$

$$+\frac{k_{17}}{4}\Bigl[t^2\Bigl(\frac{2\omega_o}{\omega_o^2-1}\cos t - \frac{2}{\omega_o^2-1}\sin t + \frac{1}{\omega_o+1}\sin(2\omega_o+1)t$$

$$+\frac{1}{\omega_o-1}\sin(2\omega_o-1)t + \frac{1}{\omega_o+1}\cos(2\omega_o+1)t + \frac{1}{\omega_o-1}\cos(2\omega_o-1)t\Bigr)$$

$$+4t\Bigl(\frac{1}{(\omega_o-1)^2} - \frac{1}{(\omega_o+1)^2}\Bigr)\sin t + \frac{2}{(\omega_o+1)^3}\Bigl(\sin t - \sin(2\omega_o+1)t$$

$$-\cos t - \cos(2\omega_0+1)t + 2\sin\omega_0 t + 2\cos\omega_0 t\Big)$$

$$-\frac{2}{(\omega_0-1)^3}\Big(\sin t + \sin(2\omega_0-1)t + \cos t + \cos(2\omega_0-1)t - 2\sin\omega_0 t$$

$$-2\cos\omega_0 t\Big)\Big] - \frac{3}{\omega_0-1}\Big[\Big(\frac{1}{(3\omega_0-1)^2} - \frac{1}{(\omega_0-1)^2}\Big)\cos(2\omega_0-1)t$$

$$+\Big(\frac{1}{3\omega_0-1} - \frac{1}{\omega_0-1}\Big)t\sin(2\omega_0-1)t + \Big(\frac{1}{(\omega_0-1)^2} - \frac{1}{(3\omega_0-1)^2}\Big)\cos\omega_0 t\Big]$$

$$\frac{-3}{(\omega_0-1)^2}\Big[\frac{1}{3\omega_0-1}\cos(2\omega_0-1)t - \frac{1}{\omega_0-1}\cos t + \Big(\frac{1}{\omega_0-1} - \frac{1}{3\omega_0-1}\Big)\cos\omega_0 t\Big]\Big\} \quad (C-3)$$

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13. ABSTRACT

Several techniques for finding approximate solutions to certain classes of nonlinear differential equations are investigated. The nonlinear systems evaluated are second order with quadratic and cubic terms, having driving functions and initial conditions. Pipes' technique is used to reduce the nonlinear differential equation to a system of linear differential equations. The Brady-Baycura technique makes use of nonlinear Laplace transforms to obtain the solution. The solutions are compared to the well known Runge-Kutta numerical method solution using the digital computer. Greater accuracy was found using the Brady-Baycura method, but the simplicity of Pipes' method makes it more attractive to the engineer.

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